JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 102, No. 2, pp. 263-276, AUGUST 1999

Controllability of the Vallée-Poussin Problem for Impulsive Differential Systems[1]

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Communicated by L. D. Berkovitz

Abstract. Necessary and sufficient conditions for the controllability of multipoint boundary-value problems for linear semihomogeneous and nonhomogeneous impulsive differential systems are established. The set of all controls solving such problems is described. Moreover, we provide sufficient conditions for the time-optimal control of the Vallée-Poussin problem and obtain the Pontryagin maximum principle in sufficient form.

Key Words. Impulsive differential systems, controls, Vallée-Poussin problem, linear systems, Pontryagin maximum principle.

1. Introduction and Preliminaries

The controllability problem of linear and quasilinear systems of ordinary differential equations has been the subject of intensive investigations during the last several decades (see for instance Refs. 1-4 and references cited therein). Recently, there has been an increasing interest in studying differential equations with impulse effect (Refs. 5-8). This is mainly due to the fact that many real-life problems are better described by impulsive differential systems. We note here that impulse action can also be used in optimal control problems (Ref. 9).

The present paper is concerned with the controllability of multipoint boundary-value problems for linear impulsive systems. Necessary and

[1]The authors are grateful to the referees for their valuable suggestions and comments to improve the content of this paper. The first author is thankful to the Scientific and Technical Research Council of Turkey (TÜBİTAK) for their financial support.

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sufficient conditions are established for the controllability of multipoint boundary-value problems for linear semihomogeneous and nonhomogeneous impulsive differential systems. We also give sufficient conditions for the time-optimal control of the Vallée–Poussin problem.

Let α and β be fixed real numbers such that $\alpha < \beta$, and let r and p be fixed positive integers. Denote by $L_2^r[\alpha, \beta]$ the set of all square integrable functions η on $[\alpha, \beta]$ with range in R^r, and denote by $D^r[1, p]$ the set of all finite sequences $\{\xi_i\}$, $\xi_i \in R^r$, $i = \overline{1, p}$. We define a space $\Pi_p^r = L_2^r \times D^r$, denote its elements by $\{\eta, \xi\}$, and let

$$\langle \{\eta, \xi\}, \{w, v\} \rangle = \int_\alpha^\beta (\eta, w)\, dt + \sum_{i=1}^{p} (\xi_i, v_i)$$

be an inner product in Π_p^r, where $(\ ,\ )$ is the Euclidean scalar product in R^r. For a function $\eta: [\alpha, \beta] \to R^r$, we use the notation $\{\eta, \eta\}$ to denote $\{\eta(t), \eta(\theta_i)\} \in \Pi_p^r$, where $\{\theta_i\}$ is a sequence of real numbers to be defined.

We shall consider the nonhomogeneous multipoint boundary-value problem

$$dx/dt = A(t)x(t) + C(t)u + f(t), \qquad t \neq \theta_i, \tag{1a}$$

$$\Delta x|_{t=\theta_i} \equiv x(\theta_i +) - x(\theta_i) = B_i x + D_i v_i + J_i, \tag{1b}$$

$$x(\alpha_k) = a_k, \qquad k = \overline{1, q}, \qquad \alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_q = \beta, \tag{2}$$

and will refer to it as the control problem (γ_1). In the special case when $a_k = 0$, $k = \overline{1, q}$, we will denote this problem by (γ_2).

With regard to Eqs. (1), the following conditions are assumed to hold: $x \in R^n$; A and C are matrix functions of sizes $n \times n$ and $n \times m$, whose elements belong to $L_2^1[\alpha, \beta]$; B_i and D_i are $n \times n$ and $n \times m$ constant matrices; $\det(I + B_i) \neq 0$, $i = \overline{1, p}$; $\{f, J\} \in \Pi_p^n[\alpha, \beta]$; and $\{\theta_i\}$, $\theta_i \in R$, $i = \overline{1, p}$, is a strictly increasing sequence in (α, β).

We note that solutions of (1) are left continuous in their domain of definitions and absolutely continuous on each of the intervals $[\alpha, \theta_1]$, $(\theta_i, \theta_{i+1}]$, $i = \overline{1, p-1}$, and $(\theta_p, \beta]$.

The control problem (γ_1) is said to be solvable if, for each $\{f, J\} \in \Pi_p^n$ and every $a_k \in R^n$, $k = \overline{1, q}$, there exists a control $\{u, v\} \in \Pi_p^m$ such that problem (1) and (2) has a solution.

The following lemma indicates that problems (γ_1) and (γ_2) are equivalent.

Lemma 1.1. Problem (γ_1) is solvable if and only if problem (γ_2) is.

Proof. Let problem (γ_1) be solvable. Since (γ_2) is a particular case of (γ_1), problem (γ_2) is also solvable.

Now, suppose, that problem (γ_2) is solvable. Let $\phi(t)$ be the Lagrange polynomial such that

$$\phi(\alpha_k)=a_k, \qquad \text{for } k=\overline{1,q}.$$

Replacing $x(t)$ by $y(t)+\phi(t)$ in (1), we see that $y(t)$ satisfies

$$dy/dt=A(t)y(t)+C(t)u+[A(t)\phi(t)-\phi'(t)+f(t)], \qquad t\neq\theta_i,$$

$$\Delta y|_{t=\theta_i}=B_iy+D_iv_i+[J_i+B_i\phi(\theta_i)],$$

$$y(\alpha_k)=0, \qquad k=\overline{1,q}, \qquad \alpha=\alpha_1<\alpha_2<\cdots<\alpha_q=\beta.$$

This problem is solvable due to our assumption. □

Associated with (1), consider the linear homogeneous system

$$dx/dt=A(t)x(t), \qquad t\neq\theta_i,$$

$$\Delta x|_{t=\theta_i}=B_ix,$$

and its adjoint problem (see Ref. 5), given by

$$dz/dt=-A^T(t)z(t), \qquad t\neq\theta_i, \tag{3a}$$

$$\Delta z|_{t=\theta_i}=-(I+B_i^T)^{-1}B_i^Tz, \tag{3b}$$

where T denotes transposition.

Let $Y(t)$ be the fundamental matrix solution of the system (3). It is clear that $Y(t)$ is left continuous, piecewise absolutely continuous, and has discontinuities of the first kind at times $t=\theta_i$, $i=\overline{1,p}$.

For each k, $k=\overline{1,q-1}$, we construct

$$Z_k(t)=\begin{cases} Y(t), & \text{if } t\in[\alpha,\alpha_{k+1}],\\ 0, & \text{otherwise,}\end{cases}$$

and define the matrix

$$Z(t)=[Z_1(t),Z_2(t),Z_3(t),\ldots,Z_{q-1}(t)].$$

One can verify easily that each solution of (3) having discontinuities at times $t=\alpha_k$ can be written in the form

$$z(t)=Z(t)c,$$

where $c\in R^{n(q-1)}$ is a constant vector. Therefore, $Z(t)$ can also be called a fundamental matrix solution of (3).

2. The Vallée–Poussin Problem

In this section, we deal with multipoint boundary-value problems containing no control.

First, we are concerned with the solvability of semihomogeneous multipoint boundary-value problems of the form

$$dx/dt = A(t)x(t) + F(t), \qquad t \neq \theta_i, \tag{4a}$$

$$\Delta x|_{t=\theta_i} = B_i x + W_i, \tag{4b}$$

$$x(\alpha_k) = 0, \qquad k = \overline{1, q}, \qquad \alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_q = \beta, \tag{5}$$

where $\{F, W\} \in \Pi_p^n$.

Theorem 2.1. The semihomogeneous Vallée–Poussin problem (4) and (5) is solvable if and only if the element $\{F, W\} \in \Pi_p^n$ is orthogonal in Π_p^n to each column of the matrix $Z(t)$.

Proof. Consider the two-point semihomogeneous boundary-value problem

$$dx/dt = A(t)x(t) + F(t), \qquad t \neq \theta_i, \tag{6a}$$

$$\Delta x|_{t=\theta_i} = B_i x + W_i, \tag{6b}$$

$$x(\alpha) = x(\beta) = 0. \tag{7}$$

We shall show that this problem is solvable if and only if each solution $z(t)$ of the system (3) satisfies the condition

$$\langle \{F, W\}, \{z, z\} \rangle = 0. \tag{8}$$

Let $X(t)$ be the fundamental matrix solution of the system (4) satisfying $X(\alpha) = I$. It is known (see Ref. 5) that solutions of (6) are of the form

$$x(t) = X(t)x(\alpha) + \int_\alpha^t X(t)X^{-1}(s)F(s)\, ds + \sum_{\alpha < \theta_i < t} X(t)X^{-1}(\theta_i t) W_i. \tag{9}$$

Making use of the boundary condition (7) and the equality

$$Y^T(t)X(t) = M^{-1},$$

where M is a constant matrix, it follows from (9) that

$$M\left[\int_\alpha^\beta Y^T(t)F(t)\, dt + \sum_{i=1}^p Y^T(\theta_i t) W_i\right] = 0. \tag{10}$$

Clearly, we can conclude from (10) that, for each i, $i=\overline{1,n}$,

$$\langle \{F, W\}, \{y_i, y_i\} \rangle = 0, \tag{11}$$

where y_i is the ith column of $Y(t)$. Since (11) implies (8), necessity is proved.

To prove sufficiency, assume that condition (8) holds. Then, it is clear that (11) is true, and hence (10) is satisfied. Now, it follows that

$$\int_\alpha^\beta X(t)X^{-1}(s)F(s)\,ds + \sum_{\alpha<\theta_i<\beta} X(t)X^{-1}(\theta_i t)W_i = 0. \tag{12}$$

In view of (12), we see that the solution $x(t), x(\alpha)=0$, satisfies also $x(\beta)=0$. By using the definition of inner product of $\Pi_p^r[\alpha_k, \alpha_{k+1}]$, and by repeating the above argument for each segment $[\alpha_k, \alpha_{k+1}]$, $k=\overline{1, q-1}$, we complete the proof of the theorem. □

Corollary 2.1. The semihomogeneous Vallée–Poussin problem (4) and (5) is solvable if and only if

$$\int_{\alpha_{k-1}}^{\alpha_k} Y^T(t)F(t)\,dt + \sum_{\alpha_{k-1}\le\theta_i\le\alpha_k} Y^T(\theta_i)W_i = 0, \qquad k=\overline{2,q}.$$

Now, let us consider the nonhomogeneous multipoint boundary-value problem (4), (2). By making use of the substitution

$$x(t) = y(t) + \phi(t)$$

as in the proof of Lemma 1.1 and, in addition, imposing

$$\phi(\theta_i)=0, \qquad \text{for } i=\overline{1,p},$$

with the help of Theorem 2.1 we find that the Vallée–Poussin problem (4), (2) is solvable if and only if

$$\int_\alpha^\beta Z^T(t)[F(t)-(\phi'(t)-A(t)\phi(t))]\,dt + \sum_{i=1}^p Z^T(\theta_i)W_i = 0. \tag{13}$$

Employing the integration by parts formula in (13), we obtain the following theorem.

Theorem 2.2. The Vallée–Poussin problem (4), (2) is solvable if and only if

$$\int_\alpha^\beta Z^T(t)F(t)\,dt + \sum_{i=1}^p Z^T(\theta_i)W_i$$
$$= Z^T(\beta)a_q - Z^T(\alpha)a_1 + \sum_{i=2}^{q-1} [Z^T(\alpha_i-) - Z^T(\alpha_i+1)]a_i.$$

In view of Corollary 2.1, we can also see that the following theorem holds.

Theorem 2.3. The Vallée-Poussin problem (4), (2) is solvable if and only if, for $k=\overline{2, q}$,

$$\int_{\alpha_{k-1}}^{\alpha_k} Y^T(t)F(t)\,dt + \sum_{\alpha_{k-1}\leq\theta_i<\alpha_k} Y^T(\theta_i)W_i = Y^T(\alpha_k)a_k - Y^T(\alpha_{k-1})a_{k-1}. \quad (14)$$

3. Solvability of Problems

We are now in a position to state and prove theorems which provide necessary and sufficient conditions for the controllability of the Vallée–Poussin problem for linear semihomogeneous and nonhomogeneous impulsive differential systems.

Theorem 3.1. Problem (γ_1) is solvable if and only if

$$\langle\{Cu, D_iv_i\}, \{z, z\}\rangle = 0, \qquad \forall\{u, v\}\in\Pi_p^m, \quad (15)$$

is satisfied only by the trivial solution z of (3).

Proof. Sufficiency. Let

$$Z(t) = [z_1(t), z_2(t), \dots, z_{n(q-1)}(t)]$$

be a fundamental matrix of (3), $c\in R^{n(q-1)}$. According to the conditions of the theorem, the infinite system

$$\langle\{Cu, D_iv_i\}, \{Zc, Zc\}\rangle = 0, \qquad \forall\{u, v\}\in\Pi_p^m,$$

has only the trivial solution $c=0$.

We first show that there are $n(q-1)$ elements $\{u^k, v^k\}\in\Pi_p^m$ such that the matrix N with elements

$$N_{jk} = \langle\{Cu^k, Dv^k\}, \{z_j, z_j\}\rangle, \qquad j, k=\overline{1, n(q-1)},$$

is nonsingular. Suppose on the contrary that N is singular, and denote by $c=c_0$ a nontrivial solution of

$$\langle\{Cu^k, Dv^k\}, \{Zc, Zc\}\rangle = 0, \qquad k=\overline{1, n(q-1)-1}.$$

Without loss of generality, we may assume that the last row of N can be written as a linear combination of the other rows. Therefore, it is true that, for any $\{u, v\}$, there exist real numbers μ_k, $k=\overline{1, n(q-1)-1}$, such that

$$\langle\{Cu, Dv\}, \{z_j, z_j\}\rangle = \sum_{k=1}^{n(q-1)-1} \mu_k \langle\{Cu^k, Dv^k\}, \{z_j, z_j\}\rangle,$$

$$j=\overline{1, n(q-1)}. \tag{16}$$

Let

$$c_0 = \begin{bmatrix} c_{01} \\ c_{02} \\ \vdots \\ c_{0n(q-1)} \end{bmatrix}, \qquad c_{oi} \in R, \qquad i=\overline{1, n(q-1)}.$$

Using (16), we have

$$\begin{aligned}
&\langle\{Cu, Dv\}, \{Z(t)c_0, Z(t)c_0\}\rangle \\
&= \left\langle \{Cu, Dv\}, \left\{ \sum_{j=1}^{n(q-1)} z_j(t)c_{0j}, \sum_{j=1}^{n(q-1)} z_j(t)c_{0j} \right\} \right\rangle \\
&= \sum_{j=1}^{n(q-1)} c_{0j} \langle\{Cu, Dv\}, \{z_j, z_j\}\rangle \\
&= \sum_{j=1}^{n(q-1)} c_{0j} \sum_{k=1}^{n(q-1)-1} \mu_k \langle\{Cu^k, Dv^k\}, \{z_j, z_j\}\rangle \\
&= \sum_{k=1}^{n(q-1)-1} \mu_k \sum_{j=1}^{n(q-1)} c_{0j} \langle\{Cu^k, Dv^k\}, \{z_j, z_j\}\rangle \\
&= \sum_{k=1}^{n(q-1)-1} \mu_k \langle\{Cu^k, Dv^k\}, \{Z(t)c_0, Z(t)c_0\}\rangle \\
&= \sum_{k=1}^{n(q-1)-1} \mu_k \times 0 = 0.
\end{aligned}$$

This means that $Z(t)c_0$, a nontrivial solution of (3), satisfies (15). Consequently, the matrix N is nonsingular.

Now, consider the multipoint boundary-value problem

$$dx/dt = A(t)x(t) + f(t) - C(t) \sum_{k=1}^{n(q-1)} c_k u^k(t), \qquad t \neq \theta_i, \tag{17a}$$

$$\Delta x|_{t=\theta_i} = B_i x + J_i - D_i \sum_{k=1}^{n(q-1)} c_k v_i^k, \tag{17b}$$

$$x(a_k) = 0, \qquad k=\overline{1, q}, \tag{17c}$$

where $\{u^k, v^k\} \in \Pi_p^m$ for $k=\overline{1, n(q-1)}$ are defined as above. Since the matrix N is nonsingular, the system

$$\sum_{k=1}^{n(q-1)} \langle \{Cu^k, Dv^k\}, \{z_j, z_j\} \rangle c_k = \langle \{z_j, z_j\}, \{f, J\} \rangle, \qquad j=\overline{1, n(q-1)},$$

has a solution; therefore, according to Theorem 2.2, (17) is a solvable problem.

Necessity. Suppose that problem (γ_1) is a solvable problem and there is a nontrivial solution z of (3) which satisfies (15). It is easy to show that there exists $\{f, J\} \in \Pi_p^n$ which satisfies $\langle \{z, z\}, \{f, J\} \rangle \neq 0$. If we add this inequality to the expression (15), then we see easily that the pair $\{f, J\}$ also satisfies the relation

$$\langle \{z, z\}, \{Cu+f, Dv+J\} \rangle \neq 0, \qquad \text{for all } \{u, v\}.$$

But this contradicts Theorem 2.1 about the existence of a solution of the Vallée–Poussin problem (4), (2), and therefore completes the proof. □

The theorem just proved implies that problems (γ_1) and (γ_2) are solvable if and only if the system

$$\langle \{Cu, Dv\}, \{Zc, Zc\} \rangle = 0, \qquad \forall \{u, v\} \in \Pi_p^m, \tag{18}$$

has only a trivial solution with respect to $c \in R^{n(q-1)}$.

Letting c be a solution of (18) and substituting

$$\{u, v\} = \{C^T Zc, D^T Zc\}$$

into (18), we see that

$$\langle \{C^T Zc, D^T Zc\}, \{C^T Zc, D^T Zc\} \rangle = 0,$$

and thus obtain the following theorem.

Theorem 3.2. Problems (γ_1) and (γ_2) are solvable if and only if the system

$$C^T(t)Z(t)c = 0, \qquad t \in [\alpha, \beta], \qquad D_i^T Z(\theta_i)c = 0, \qquad i=\overline{1,p}, \tag{19}$$

has only the trivial solution.

Defining

$$Q(t) = Z^T(t)C(t), \qquad P_i = Z^T(\theta_i)D_i,$$

and setting

$$\Gamma = \int_\alpha^\beta Q(t)Q^T(t)\,dt + \sum_{i=1}^{p} P_i P_i^T,$$

in view of (19), we see that $\Gamma c = 0$, and formulate it as follows.

Theorem 3.3. Problems (γ_1) and (γ_2) are solvable if and only if the Gram matrix Γ is nonsingular.

4. Set of Solving Controls

In view of Theorems 2.2 and 2.3, it is not difficult to see that the following theorem is true.

Theorem 4.1. The control $\{u, v\} \in \Pi_p^m$ solves problem (γ_1) if and only if

$$\begin{aligned}&\int_\alpha^\beta Z^T(t)[f(t) + C(t)u(t)]\,dt + \sum_{i=1}^{p} Z^T(\theta_i)[J_i + D_i v_i]\\ &= \sum_{i=2}^{q-1} [Z^T(\alpha_i -) - Z^T(\alpha_i +)]a_i + Z^T(\beta)a_q - Z^T(\alpha)a_1,\end{aligned} \tag{20}$$

or for $k = \overline{2, q}$,

$$\begin{aligned}&\int_{\alpha_{k-1}}^{\alpha_k} Y^T(t)[f(t) + C(t)u(t)]\,dt + \sum_{\alpha_{k-1} \le \theta_i < \alpha_k} Y^T(\theta_i)[J_i + D_i v_i]\\ &= Y^T(\alpha_k)a_k - Y^T(\alpha_{k-1})a_{k-1}.\end{aligned} \tag{21}$$

Letting

$$\begin{aligned}K = \Gamma^{-1}\bigg\{&\int_\alpha^\beta Z^T(t)f(t)\,dt + \sum_{i=1}^{p} Z^T(\theta_i)J_i\\ &+ \sum_{i=2}^{q-1} [Z^T(\alpha_i +) - Z^T(\alpha_i -)]a_i + Z^T(\alpha_1)a_1 - Z^T(\beta)a_q\bigg\},\end{aligned}$$

and taking Theorem 2.3 into account, we see easily that the following theorem is also true.

Theorem 4.2. Suppose that problem (γ_1) is solvable. Then, the control $\{u, V\}$ with $U = Q^T(t)K$ and $V_i = P_i^T K$ solves problem (γ_1).

The control $\{U, V\}$ gives us a tool to describe the set of all controls solving problem (γ_1).

Theorem 4.3. A control $\{u, v\}$ solves problem (γ_1) if and only if it is of the form

$$u = U + \hat{u}, \qquad v = V + \hat{v}, \tag{22}$$

where $\{\hat{u}, \hat{v}\} \in \Pi_p^m$ is orthogonal in Π_p^m to all the columns of the matrix $\{Q^T(t), P_i^T\}$.

Proof. Indeed, let $\{u, v\}$ be a control for problem (γ_1). Then,

$$\int_\alpha^\beta Q(t)(u(t) - U(t))\, dt + \sum_{i=1}^p P_i(v_i - V_i) = 0.$$

If we set

$$\hat{u} = u - U, \qquad \hat{v} = v - V,$$

then we obtain the necessity part of the theorem.

Conversely, suppose that (22) is satisfied. Then, condition (20) is valid; therefore, the control $\{u, v\}$ solves problem (γ_1). □

It can be shown (see Ref. 3) that the norm $\|\{U, V\}\|_m$, where

$$\|\{u, v\}\|_m = \langle\{u, v\}, \{u, v\}\rangle^{1/2},$$

is the smallest among all controls solving problem (γ_1).

5. Time-Optimal Control for the Vallée–Poussin Problem

For each fixed $\tau \in [\alpha, \infty)$, define

$$\Pi^m[\alpha, \tau] = L_2^m[\alpha, \tau] \times D^m[1, p_\tau],$$

and let $\{\theta_i\}$ be a strictly increasing sequence of real numbers in $[\alpha, \infty)$, $\alpha < \theta_1 < \theta_2 < \cdots < \theta_{p_\tau} < \tau$. Consider the system

$$dx/dt = A(t)x(t) + C(t)u + f(t), \qquad t \neq \theta_i, \tag{23a}$$

$$\Delta x|_{t=\theta_i} = B_i x + D_i v_i + J_i, \tag{23b}$$

$$x(\alpha_k) = a_k, \qquad k = \overline{0, q}, \qquad 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_q = \tau, \tag{24}$$

where $\chi \in R^n$, the points $a_k \in R^n$, $k = \overline{0, q}$, and the initial time $\alpha_0 = 0$ are fixed. The times $\alpha_k \in R$, $k = \overline{1, q}$, with

$$\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_q = \tau,$$

are unknown. We assume that the elements of A and C belong to $L_2^1[\alpha, \tau]$ and $\{f, J\} \in \Pi^n[\alpha, \tau]$ for every $\tau > 0$.

We say that the control problem (23) and (24), which is denoted by problem (γ_3), is solvable if, for a given set of arbitrary points $a_0, a_1, \ldots, a_q$ from R^n, there exist times $\alpha_1, \alpha_2, \ldots, \alpha_q$ and a control $\{\hat{u}, \hat{v}\} \in \Pi_p^m$ such that the system

$$dx/dt = A(t)x(t) + C(t)\hat{u} + f(t), \qquad t \neq \theta_i,$$

$$\Delta x|_{t=\theta_i} = B_i x + D_i \hat{v}_i + J_i,$$

has a solution satisfying (24).

Let $\Delta_1 \times \Delta_2$ be a bounded subset of $R^m \times R^m$. The control $\{\hat{u}, \hat{v}\}$ such that $\hat{u} \in \Delta_1$ and $\hat{v}_i \in \Delta_2$ for all $t > \alpha$ and $i > 1$ is called the time-optimal control if it solves problem (γ_3) in the least possible τ.

Let us consider the simple case $q = 1$. We say that the control $\{\hat{u}, \hat{v}\}$ with vector $c = c_0$ from R^n satisfies the Pontryagin condition in the domain $\Delta_1 \times \Delta_2$ and on the segment $[\alpha, \tau]$, where $\tau > \alpha$ is a fixed real number, if $c_0^T Y^T(t) C(t) u(t)$ takes on its maximum value for almost all $t \in [\alpha, \tau]$ when $u = \hat{u}$, and if $c_0^T Y^T(\theta_i) D_i v_i$ takes on its maximum value for all i such that $\alpha < \theta_i < \tau$ when $v_i = \hat{v}_i$. Furthermore, it is necessary to check that

$$c_0^T Y^T(t)[C(t)u(t) + f(t) + A(t)a_2] > 0, \qquad \text{for almost all } t \in [\alpha, \tau],$$

and that

$$c_0^T Y^T(\theta_i)[D_i v_i + J_i + B_i(I + B_i)^{-1} a_2] > 0, \qquad \alpha < \theta_i < \tau.$$

Theorem 5.1. Let $\{\hat{u}, \hat{v}\}$ be a control which solves problem (γ_3) with $q = 1$ at time $\tau > \alpha$ and satisfies the Pontryagin condition in the domain $\Delta_1 \times \Delta_2$ and on the segment $[\alpha, \tau]$ with some vector $c = c_0$. Then, $\{\hat{u}, \hat{v}\}$ and the corresponding trajectory are time optimal.

Proof. According to (21), we can write

$$\int_\alpha^\tau Y^T(t)[f(t) + C(t)\hat{u}(t)]\, dt + \sum_{\alpha < \theta_i < \tau} Y^T(\theta_i)[J_i + D_i \hat{v}_i]$$
$$= Y^T(\tau) a_1 - Y^T(\alpha) a_0. \tag{25}$$

It is well-known that a fundamental matrix $Y(t)$ satisfies

$$Y(\tau) = Y(\alpha) - \int_\alpha^\tau A^T(t) Y(t)\, dt - \sum_{\alpha < \theta_i < \tau} (I + B_i^T)^{-1} B_i^T Y(\theta_i).$$

Substituting this expression with $Y(\alpha)=I$ into (25), we obtain

$$\int_{\alpha}^{\tau} Y^T(t)[f(t)+C(t)\hat{u}(t)+A(t)a_1]\,dt + \sum_{\alpha<\theta_i<\tau} Y^T(\theta_i)[J_i+D_i\hat{v}_i+B_i(I+B_i)^{-1}a_1]=a_1-a_0. \tag{26}$$

Suppose on the contrary that there is a control $\{u, v\}$ taking $x(\alpha)=a_0$ to $x(\theta)=a_1$ at time $\theta<\tau$. Then,

$$\int_{\alpha}^{\theta} c_0^T Y^T(t)[f(t)+C(t)u(t)+A(t)a_1]\,dt + \sum_{\alpha<\theta_i<\theta} c_0^T Y^T(\theta_i)[J_i+D_iv_i+B_i(I+B_i)^{-1}a_1]=c_0^T(a_1-a_0). \tag{27}$$

Multiplying (26) from the left by c_0^T and subtracting (27) from the resulting equation, we find that

$$\int_{\alpha}^{\theta} c_0^T Y^T(t)C(t)[\hat{u}(t)-u(t)]\,dt + \sum_{\alpha<\theta_i<\theta} c_0^T Y^T(\theta_i)D_i[\hat{v}_i-v_i]$$
$$+\int_{\theta}^{\tau} c_0^T Y^T(t)[f(t)+C(t)\hat{u}(t)+A(t)a_1]\,dt$$
$$+\sum_{\theta\le\theta_i<\tau} c_0^T Y^T(\theta_i)[J_i+D_i\hat{v}_i+B_i(I+B_i)^{-1}a_1]=0. \tag{28}$$

According to our assumptions, the first and second terms in (28) are nonnegative, and the third and fourth terms in (28) are positive. This contradiction completes the proof. □

For $k=\overline{1, q}$, we construct the matrices

$$Z_k(t)=\begin{cases} Y(t), & t\in[0, \alpha], \\ 0, & \text{otherwise,}\end{cases}$$

and define

$$Z(t)=[Z_1(t), Z_2(t), \dots, Z_q(t)].$$

Let $\{\hat{u}, \hat{v}\}$ be a control which solves problem (γ_3) at time $\tau>\alpha$. We say that this control with a vector $c=c_0$ satisfies the Pontryagin condition in the domain $\Delta_1\times\Delta_2$ and on the segment $[\alpha, \tau]$ if, for $k=\overline{1, q}$,

$$c_0^T Z^T(t)C(t)\hat{u}(t)=\max_{u\in\Delta_1} c_0^T Z^T(t)C(t)u, \qquad \text{for almost all } t\in[\alpha, \tau],$$

$$c_0^T Z^T(t)[C(t)\hat{u}(t)+f(t)+A(t)a_k]>0, \qquad \text{for almost all } t\in[\alpha_{k-1}, \alpha_k],$$

and

$$c_0^T Z^T(\theta_i) D_i \hat{v}_i = \max_{v \in \Delta_2} c_0^T Z^T(\theta_i) D_i v_i, \qquad \alpha < \theta_i < \tau,$$

$$c_0^T Z^T(\theta_i)[J_i + D_i \hat{v}_i + B_i (I + B_i)^{-1} a_k] > 0, \qquad \alpha_{k-1} \leq \theta_i < \alpha_k.$$

The theorem below is obtained from Theorem 5.1.

Theorem 5.2. Let $\{\hat{u}, \hat{v}\}$ be a control which solves problem (γ_3) at time $\tau > \alpha$ and with a vector $c_0 \in R^{n(q-1)}$ satisfies the Pontryagin condition in the domain $\Delta_1 \times \Delta_2$ and on the segment $[\alpha, \tau]$. Then, $\{\hat{u}, \hat{v}\}$ and the corresponding trajectory are time optimal.

Corollary 5.1. Let the conditions of Theorem 5.2 be valid, and let $x = x(t)$, $\alpha \leq t \leq \tau$, be the time-optimal trajectory connecting the points $a_0 = x(\alpha)$ and $a_q = x(\tau)$. Then, each part of this trajectory connecting the points $x(\omega)$ and $x(\nu)$, where $\alpha < \omega < \nu < \tau$, is also a time-optimal trajectory.

We end this paper with a theorem which provides a criterion analogue to that of Kalman for problem (γ_3).

Theorem 5.3. Assume that A and C are constant matrices, $B_i \equiv 0$, and $D_i \equiv D$. Then, problem (γ_3) is solvable if and only if the rank of the matrix $[C, D, AC, AD, \dots, A^{n-1}C, A^{n-1}D]$ is n.

Proof. It follows from (19) that

$$C^T \exp(-A^T t) c = 0, \qquad D^T \exp(-A^T \theta_i) c = 0, \qquad \text{for all } t \text{ and } i. \tag{29}$$

Since (29) is equivalent to the infinite system

$$C^T c = 0,\ C^T A^T c = 0, \dots, C^T (A^T)^k c = 0, \dots,$$

$$D^T c = 0,\ D^T A^T c = 0, \dots, D^T (A^T)^k c = 0, \dots,$$

by making use of the Cayley-Hamilton theorem and arguments developed in Ref. 3, we easily complete the proof. □

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