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Boundary value problems for higher order linear impulsive differential equations

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Abstract

In this paper higher order linear impulsive differential equations with fixed moments of impulses subject to linear boundary conditions are studied. Green's formula is defined for piecewise differentiable functions. Properties of Green's functions for higher order impulsive boundary value problems are introduced. An appropriate example of the Green's function for a boundary value problem is provided. Furthermore, eigenvalue problems and basic properties of eigensolutions are considered.

Keywords: Impulsive differential equations; Boundary value problems; Eigenvalue problems; Green's function

1. Introduction

The problem in defining a higher order impulsive differential equation is, basically, that a function having a discontinuity does not possess a derivative. To deal with this situation, one may consider the difference of one-sided derivatives at such a point of discontinuity.

Let $J = [\alpha, \beta] \subset \mathbb{R}$ be a closed interval and $\langle\theta_i\rangle_{i=1}^{p} \subset J$ be a finite sequence of impulse points θ_i such that

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_p < \theta_{p+1} = \beta. \tag{1.1}$$

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Let $\mathcal{PLC} = \mathcal{PLC}(J, \theta_i; \mathbb{F})$ denote the set of all functions $f : J \to \mathbb{F}$ that are left continuous for all $t \in J$, and have finite jumps at $t = \theta_i$. Here $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Similarly, define $\mathcal{PLC}^n = \mathcal{PLC}^n(J, \theta_i; \mathbb{F})$ as follows:

$$\mathcal{PLC}^n = \left\{ f \in \mathcal{PLC}\colon\ f \in C^n\left(J \setminus \langle\theta_i\rangle_{i=1}^p\right) \text{ such that } \Delta f^{(n)}\big|_{t=\theta_i} < \infty \right\}, \tag{1.2}$$

where $C^n(J \setminus \langle\theta_i\rangle_{i=1}^p)$ is the set of functions that are n times continuously differentiable in $J \setminus \langle\theta_i\rangle_{i=1}^p$, and the jump

$$\Delta f^{(n)}\big|_{t=\theta_i} = f^{(n)}\left(\theta_i^+\right) - f^{(n)}\left(\theta_i^-\right) \tag{1.3}$$

is defined as the difference of the limits

$$f^{(n)}\left(\theta_i^\pm\right) = \lim_{h\to 0^\pm} f^{(n)}(\theta_i + h) \tag{1.4}$$

at the impulse points $t = \theta_i$.

We consider an nth order linear impulsive differential equation of the form

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), & t \neq \theta_i, \\ \Delta x^{(j-1)}\big|_{t=\theta_i} - \sum_{k=1}^n b_{ijk}x^{(k-1)}(\theta_i^-) = a_{ij}, & i = 1, \ldots, p,\ j = 1, \ldots, n, \end{cases} \tag{1.5}$$

where the functions $p_0, \ldots, p_n$ and f are assumed to be in $\mathcal{PLC}$, and the coefficients b_{ijk} and a_{ij} are in $\mathbb{F}$. In order to simplify the notation, we rewrite (1.5) as

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = f(t), & t \neq \theta_i, \\ \Delta \hat{x}|_{t=\theta_i} - B_i\hat{x}(\theta_i^-) = a_i, & i = 1, \ldots, p, \end{cases} \tag{1.6}$$

where $\hat{x}(t) = [x(t), \ldots, x^{(n-1)}(t)]^T$ for $t \neq \theta_i$ and $\hat{x}(\theta_i^\pm) = \lim_{h\to 0^\pm} \hat{x}(\theta_i + h)$ for every $i = 1, \ldots, p$, and

$$B_i = \begin{pmatrix} b_{i11} & \cdots & b_{i1n} \\ \vdots & \ddots & \vdots \\ b_{in1} & \cdots & b_{inn} \end{pmatrix}, \quad a_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}, \quad i = 1, \ldots, p. \tag{1.7}$$

If $f \not\equiv 0$ or $a_i \neq 0$ for some integer i $(1 \leqslant i \leqslant p)$, then (1.6) is called *inhomogeneous*; while

$$\begin{cases} p_0(t)x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0, & t \neq \theta_i, \\ \Delta \hat{x}|_{t=\theta_i} - B_i\hat{x}(\theta_i^-) = 0, & i = 1, \ldots, p, \end{cases} \tag{1.8}$$

is called the associated *homogeneous* equation. As a matter of fact, we emphasize that the idea of considering higher order impulsive differential equations with discontinuity in all derivatives is not new [7,10–12,18,22,27]. In addition, there are many studies on boundary value problems with discontinuity conditions including periodic boundary value problems and eigenvalue problems [4–6,9,13,15,17,19,21,26,28,29].

However, we approach the problem in the general form. The generalization investigates the classical problem [8,20], and has the promise of useful application in many branches of applied mathematics [14,16,23,24].

It should be noted that the differential equation in (1.6) (or (1.8)) can also be written as a system of first order equations provided that $p_0(t) \neq 0$ for all $t \in J$. Consequently, the following theorem, in which E is the $n \times n$ identity matrix, can easily be proved.

Theorem 1. *Let $1/p_0, p_1, \dots, p_n$ and f be functions in $\mathcal{PLC}$ for any $t_0 \in J$ and $\xi = [\xi_1, \dots, \xi_n]^T \in \mathbb{F}^n$. There exists a unique solution $x(t) = x(t, t_0, \xi)$ of* (1.6)*, defined on J, satisfying the initial condition*

$$x^{(j-1)}(t_0) = \xi_j, \quad j = 1, \dots, n, \tag{1.9}$$

provided that $\det(E + B_i) \neq 0$ *for all $i = 1, \dots, p$.*

Proof. Since $1/p_0$ is assumed to be in $\mathcal{PLC}$ it follows by definition that

$$\begin{aligned} &p_0(t) \neq 0 \quad \text{for all } t \in J, \\ &p_0\big(\theta_i^+\big) \neq 0 \quad \text{for all } i = 1, \dots, p. \end{aligned} \tag{1.10}$$

Writing (1.6) as a system of first order equations the proof follows from [23, Theorem 7, p. 44]. □

In view of Theorem 1, the set of solutions of an nth order linear homogeneous impulsive differential equation (1.8) is an n-dimensional vector space over $\mathbb{F}$. Hence we may identify n linearly independent solutions $\phi_1, \dots, \phi_n$ of (1.8) as *fundamental* solutions. Furthermore, if we denote the row vector of the fundamental solutions by

$$\Phi = [\phi_1, \dots, \phi_n], \tag{1.11}$$

then the associated matrix valued function

$$\hat{\Phi} = [\widehat{\phi_1}, \dots, \widehat{\phi_n}] \tag{1.12}$$

is called a *fundamental matrix* for (1.8) and $\det \hat{\Phi}(t)$ is the *Wronskian* of $\phi_1, \dots, \phi_n$. We remark that the fundamental matrix $\hat{\Phi}$ in (1.12) is defined by

$$\hat{\Phi}(t) = \begin{pmatrix} \phi_1(t) & \cdots & \phi_n(t) \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)}(t) & \cdots & \phi_n^{(n-1)}(t) \end{pmatrix}$$

for $t \neq \theta_i$, however, at the points of discontinuity one-sided limits $\hat{\Phi}(\theta_i^{\pm})$ should be considered. Thus, using fundamental solutions, a general solution of (1.6) can be written in the form

$$x(t) = \Phi(t)c + \varphi(t), \tag{1.13}$$

where $c = [c_1, \dots, c_n]^T \in \mathbb{F}^n$ and $\varphi(t)$ is any particular solution of (1.6).

It is possible and helpful in many applications to define a particular solution of (1.6) by means of fundamental solutions of (1.8). This is achieved in the following theorem, known as the *variation of parameters formula*.

Theorem 2. *Let $\Phi(t) = [\phi_1(t), \dots, \phi_n(t)]$ be a row vector of fundamental solutions of* (1.8)*. There exists a solution $\varphi(t)$ of* (1.6) *in the form*

$$\varphi(t) = \Phi(t)\left\{ \int_{t_0}^{t} \hat{\Phi}^{-1}(s) \frac{f(s)}{p_0(s)} e_n\, ds + \sum_{t_0 \leqslant \theta_i < t} \hat{\Phi}^{-1}\big(\theta_i^+\big) a_i \right\} \tag{1.14}$$

for $t_0 \leqslant t$, *where* $e_n = [0, \dots, 0, 1]^T$ *is the nth unit vector in* $\mathbb{F}^n$. *Similarly, for* $t < t_0$, *such a solution is given by*

$$\varphi(t) = \Phi(t)\left\{\int_{t_0}^{t} \hat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}e_n\,ds - \sum_{t\leqslant\theta_i<t_0} \hat{\Phi}^{-1}\big(\theta_i^+\big)a_i\right\}. \tag{1.15}$$

Proof. It can be shown for $t_0 \leqslant t$ and $t \neq \theta_i$ that

$$\varphi^{(j)}(t) = \Phi^{(j)}(t)\left\{\int_{t_0}^{t} \hat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}e_n\,ds + \sum_{t_0\leqslant\theta_i<t} \hat{\Phi}^{-1}\big(\theta_i^+\big)a_i\right\} + \delta_{jn}\frac{f(s)}{p_0(s)},$$

where δ_{jn} is the well-known Kronecker delta such that $\delta_{jn} = 0$ for $j \neq n$ and $\delta_{nn} = 1$. Moreover, at $t = \theta_k$ for $1 \leqslant k \leqslant p$ we have

$$\varphi^{(j)}\big(\theta_k^-\big) = \Phi^{(j)}\big(\theta_k^-\big)\left\{\int_{t_0}^{t_k} \hat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}e_n\,ds + \sum_{t_0\leqslant\theta_i<\theta_k} \hat{\Phi}^{-1}\big(\theta_i^+\big)a_i\right\},$$

and, fortunately,

$$\varphi^{(j)}\big(\theta_k^+\big) = \Phi^{(j)}\big(\theta_k^+\big)\left\{\int_{t_0}^{t_k} \hat{\Phi}^{-1}(s)\frac{f(s)}{p_0(s)}e_n\,ds + \sum_{t_0\leqslant\theta_i<\theta_k} \hat{\Phi}^{-1}\big(\theta_i^+\big)a_i\right\} + a_k.$$

Thus, $\varphi(t)$ defined by Eq. (1.14) satisfies (1.6). The case $t < t_0$ can be treated similarly and this completes the proof. □

2. Green's formula

Let $\mathcal{L}$ be the differential operator of order $n \geqslant 1$ defined by

$$\mathcal{L}(x) = p_0(t)x^{(n)} + \cdots + p_n(t)x, \quad t \neq \theta_i, \tag{2.16}$$

where p_k are in $\mathcal{PLC}^{n-k}$ for $k = 0, 1, \dots, n$ and $1/p_0$ is in $\mathcal{PLC}$. Let the operators of discontinuities $\mathcal{J}_i$ be defined by

$$\mathcal{J}_i(x) = \Delta\hat{x}|_{t=\theta_i} - B_i\hat{x}\big(\theta_i^-\big), \quad i = 1, \dots, p, \tag{2.17}$$

such that $\det(E + B_i) \neq 0$ for every $i = 1, \dots, p$.

Consider the integral

$$\int_{\alpha}^{\beta} \bar{v}p_{n-k}u^{(k)}\,ds = \sum_{i=0}^{p}\int_{\theta_i}^{\theta_{i+1}} \bar{v}p_{n-k}u^{(k)}\,ds, \tag{2.18}$$

where u and v are functions in $\mathcal{PLC}^n$, and $\bar{v}$ denotes the complex conjugate of v. After a k times integration by parts applied to the integrals in (2.18) and having summed the results over k from 0 to n, we obtain

$$\int_{\alpha}^{\beta} \bar{v}\,\mathcal{L}(u)\,ds - \int_{\alpha}^{\beta} \overline{\mathcal{L}^{\dagger}(v)}\,u\,ds = \mathcal{S}(u, v)\big|_{t=\alpha}^{t=\beta} - \sum_{i=1}^{p} \Delta\mathcal{S}(u, v)|_{t=\theta_i}, \tag{2.19}$$

where the differential operator $\mathcal{L}^\dagger$, the adjoint operator for $\mathcal{L}$, is defined by

$$\mathcal{L}^\dagger(v) = (-1)^n\big(\overline{p_0(t)}v\big)^{(n)} + \cdots + \overline{p_n(t)}v, \quad t \neq \theta_i. \tag{2.20}$$

On the other hand, the bilinear form $\mathcal{S}(u, v)$ is given by

$$\mathcal{S}(u,v) = \sum_{j,k=1}^{n} \bar{v}^{(j-1)} S_{jk} u^{(k-1)} = \hat{v}^* S \hat{u}, \tag{2.21}$$

where $S = (S_{jk})$ is the $n \times n$ nonsingular matrix, and $\hat{v}^*$ denotes the conjugate transpose of $\hat{v}$. The entries S_{jk} of the matrix S can be obtained by direct calculation and are

$$S_{jk} = \begin{cases} \sum_{s=j-1}^{n-k}(-1)^s\binom{s}{j-1}p_{n-s-k}^{(s-j+1)} & \text{if } j+k \leqslant n+1, \\ (-1)^{j-1}p_0 & \text{if } j+k = n+1, \\ 0 & \text{if } j+k > n+1, \end{cases} \tag{2.22}$$

for all $j, k = 1, \ldots, n$.

It is also useful to write the jumps $\Delta\mathcal{S}(u,v)|_{t=\theta_i}$ in terms of the operators of discontinuities $\mathcal{J}_i$ as

$$\Delta\mathcal{S}(u,v)|_{t=\theta_i} = \hat{v}^*\big(\theta_i^+\big)S\big(\theta_i^+\big)\mathcal{J}_i(u) - \big(\mathcal{J}_i^\dagger(v)\big)^*\big[-S\big(\theta_i^+\big)(E+B_i)\hat{u}\big(\theta_i^-\big)\big], \tag{2.23}$$

where the adjoint operators of discontinuities $\mathcal{J}_i^\dagger$ are defined by

$$\mathcal{J}_i^\dagger(v) = \Delta\hat{v}|_{t=\theta_i} - \big\{\big[\big(E+B_i^*\big)S^*\big(\theta_i^+\big)\big]^{-1}S^*\big(\theta_i^+\big) - E\big\}\hat{v}\big(\theta_i^-\big). \tag{2.24}$$

At this point, referring to (2.24), we remark that

$$\big[\big(E+B_i^*\big)S^*\big(\theta_i^+\big)\big]^{-1}S^*\big(\theta_i^+\big) \tag{2.25}$$

is nonsingular if and only if $E + B_i$ is nonsingular. So, the conditions of Theorem 1 are clearly fulfilled whenever an impulsive differential equation formed by the operators $\mathcal{L}^\dagger$ and $\mathcal{J}_i^\dagger$ is considered.

Therefore, rewriting (2.19), using the jumps of the bilinear form $\mathcal{S}(u,v)$ at the points $t = \theta_i$ defined by (2.23), we arrive at the formula

$$\begin{aligned} &\int_\alpha^\beta \bar{v}\,\mathcal{L}(u)\,ds + \sum_{i=1}^{p}\hat{v}^*\big(\theta_i^+\big)S\big(\theta_i^+\big)\mathcal{J}_i(u) \\ &\qquad - \int_\alpha^\beta \overline{\mathcal{L}^\dagger(v)}\,u\,ds - \sum_{i=1}^{p}\big(\mathcal{J}_i^\dagger(v)\big)^*\big[-S\big(\theta_i^+\big)(E+B_i)\hat{u}\big(\theta_i^-\big)\big] \\ &\quad = \hat{v}^*(t)S(t)\hat{u}(t)\big|_{t=\alpha}^{t=\beta}, \end{aligned} \tag{2.26}$$

so that the right-hand side depends only on the boundary points α and β of the interval J. The identity (2.26) (or equivalently (2.19)) will be called *Green's formula* for functions in $\mathcal{PLC}^n$.

Moreover, let U be any boundary form of rank m defined by

$$U(x) = M\hat{x}(\alpha) + N\hat{x}(\beta), \tag{2.27}$$

where M and N are $m \times n$ matrices such that the matrix $(M : N)$, with m rows and $2n$ columns, has the property that

$$\operatorname{rank}(M : N) = m. \tag{2.28}$$

In addition, if U_c denotes any complementary boundary form of rank $2n - m$ for U then there exist, by the *boundary form formula* [8, Theorem 2.1, p. 288], unique boundary forms $U_c^\dagger$ and $U^\dagger$ of ranks m and $2n-m$, respectively, such that

$$\mathcal{S}(u,v)\big|_{t=\alpha}^{t=\beta} = \big(U_c^\dagger(v)\big)^* U(u) + \big(U^\dagger(v)\big)^* U_c(u) \tag{2.29}$$

holds for every pair of functions u and v in $\mathcal{PLC}^{n-1}$. So, the right-hand side of Green's formula (2.26) can also be written in terms of the boundary forms by using the equality in (2.29).

3. Boundary value problems

Consider the following homogeneous boundary value problem:

$$\Pi_m: \quad \begin{cases} \mathcal{L}(x) = 0, & t \neq \theta_i, \\ \mathcal{J}_i(x) = 0, & i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{3.30}$$

where U is of rank m and is defined by (2.27). The corresponding adjoint problem for Π_m has the form

$$\Pi_{2n-m}^\dagger: \quad \begin{cases} \mathcal{L}^\dagger(x) = 0, & t \neq \theta_i, \\ \mathcal{J}_i^\dagger(x) = 0, & i = 1, \ldots, p, \\ U^\dagger(x) = 0, \end{cases} \tag{3.31}$$

where $U^\dagger(x) = P^*\hat{x}(\alpha) + Q^*\hat{x}(\beta)$ is of rank $2n-m$. Note that the boundary conditions $U(x) = 0$ and $U^\dagger(x) = 0$ are adjoint if and only if

$$MS^{-1}(\alpha)P = NS^{-1}(\beta)Q \tag{3.32}$$

holds. This follows directly from (2.29). Hence, it is not difficult to conclude that the problem Π_m is self-adjoint if and only if $m = n$ and the following three conditions are satisfied:

(a) $\mathcal{L} = \mathcal{L}^\dagger$,
(b) $S^{-1}(\theta_i^+) = (E + B_i)S^{-1}(\theta_i^-)(E + B_i^*)$, $i = 1, \ldots, p$,
(c) $MS^{-1}(\alpha)M^* = NS^{-1}(\beta)N^*$.

Self-adjoint boundary value problems, such as Sturm–Liouville [11,28,29], frequently arise in models of physical systems. In the case of Sturm–Liouville boundary value problems with impulses for instance, the condition (b) can be reduced to a much simpler form by an easy calculation. A particular case of (b) has been used recently in [29] for the inverse eigenvalue problems.

Let $U\hat{\Phi}$ be defined by

$$U\hat{\Phi} = M\hat{\Phi}(\alpha) + N\hat{\Phi}(\beta), \tag{3.33}$$

where $\hat{\Phi} = [\widehat{\phi_1}, \ldots, \widehat{\phi_n}]$ is a fundamental matrix for (1.8). Namely, $\phi_1, \ldots, \phi_n$ are linearly independent solutions of the system

$$\begin{cases} \mathcal{L}(x) = 0, & t \neq \theta_i, \\ \mathcal{J}_i(x) = 0, & i = 1, \ldots, p, \end{cases} \tag{3.34}$$

defined on J. Thus for a solution $\varphi = \varphi(t)$ of (3.34) it follows that $U(\varphi) = 0$ if and only if

$$U(\hat{\Phi}c) = (U\hat{\Phi})c = 0, \tag{3.35}$$

where c is a constant vector. The following theorem gives a necessary and sufficient condition for the existence of nontrivial solutions of the problem Π_m.

Theorem 3. *The problem Π_m has exactly k, $0 \leqslant k \leqslant n$, linearly independent solutions if and only if $U\hat{\Phi}$ has rank $n-k$.*

Proof. The proof follows directly from (3.35). For, if the rank of $U\hat{\Phi}$ is $n-k$ then the number of linearly independent vectors c, satisfying (3.35), is $k = n-(n-k)$.

Moreover, if $\widehat{\Phi_1}$ is any other fundamental matrix for (3.34) then we have $\widehat{\Phi_1} = \widehat{\Phi}C$ for some nonsingular $n \times n$ matrix C. Therefore,

$$\operatorname{rank}(U\widehat{\Phi_1}) = \operatorname{rank}(U\hat{\Phi}C) = \operatorname{rank}(U\hat{\Phi})$$

completes the proof. □

According to Theorem 3, therefore, any solution $\varphi = \varphi(t)$ of the problem Π_m can be written as a linear combination of these k linearly independent solutions $\varphi_1, \dots, \varphi_k$; that is,

$$\varphi(t) = \sum_{i=1}^{k} c_i \varphi_i(t). \tag{3.36}$$

In addition, using (3.36) it is not difficult to prove the following theorem.

Theorem 4. *If Π_m has k linearly independent solutions then the adjoint problem $\Pi^{\dagger}_{2n-m}$ has $k+m-n$ linearly independent solutions.*

3.1. Inhomogeneous boundary value problems

Now, consider the linear inhomogeneous boundary value problem of rank m,

$$\begin{cases} \mathcal{L}(x) = f(t), & t \neq \theta_i, \\ \mathcal{J}_i(x) = a_i, & i = 1, \dots, p, \\ U(x) = \gamma, \end{cases} \tag{3.37}$$

where the function f is in $\mathcal{PLC}$, and a_i and γ are column vectors in $\mathbb{F}^n$. Clearly, if φ and ψ are two solutions of (3.37) then the difference, $\varphi - \psi$, is a solution of the associated homogeneous problem Π_m. Furthermore, if Π_m has k linearly independent solutions $\varphi_1, \dots, \varphi_k$, this difference can be written as a linear combination; namely,

$$\varphi - \psi = \sum_{j=1}^{k} c_j \varphi_j \tag{3.38}$$

for some constants $c_1, \dots, c_k$.

It is well known [1,2,14,23] that an inhomogeneous problem does not always possess a solution. The following theorem provides a necessary and sufficient condition for the existence of solutions of the problem (3.37).

Theorem 5. *The inhomogeneous boundary value problem* (3.37) *has a solution if and only if*

$$\int_{\alpha}^{\beta} \overline{\psi(s)} f(s)\, ds + \sum_{i=1}^{p} \hat{\psi}^*(\theta_i^+) S(\theta_i^+) a_i = \big(U_c^{\dagger}(\psi)\big)^* \gamma \tag{3.39}$$

holds for every solution ψ of the adjoint homogeneous boundary value problem $\Pi^{\dagger}_{2n-m}$.

Proof. Let φ and ψ be any solutions of Π_m and $\Pi_{2n-m}^{\dagger}$, respectively. Applying Green's formula (2.26) to functions φ and ψ, and using (2.29) proves the necessity of the condition (3.39).

Conversely, suppose condition (3.39) holds for every solution ψ of the problem $\Pi_{2n-m}^{\dagger}$. Any solution φ of the linear impulsive equation

$$\begin{cases} \mathcal{L}(x) = f(t), & t \neq \theta_i, \\ \mathcal{J}_i(x) = a_i, & i = 1, \dots, p, \end{cases} \tag{3.40}$$

can be written as

$$\varphi(t) = \Phi(t)\xi + y(t)$$

with an arbitrary constant vector ξ, where $\Phi(t)$ is a row vector of fundamental solutions of (3.34), and $y(t)$ is any particular solution of (3.40). Hence, the problem (3.37) has a solution if and only if there exists ξ so that

$$(U\hat{\Phi})\xi = \gamma - U(y) \tag{3.41}$$

holds. The system (3.41), however, has a solution ξ if and only if $\gamma - U(y)$ is orthogonal to every solution of the corresponding adjoint homogeneous system. That is, for every u satisfying

$$(U\hat{\Phi})^* u = 0, \tag{3.42}$$

we should have

$$u^*\big(\gamma - U(y)\big) = 0. \tag{3.43}$$

On the other hand, it is not difficult to see that if the problem $\Pi_{2n-m}^{\dagger}$ has k_1 linearly independent solutions $\psi_1, \dots, \psi_{k_1}$, then $U_c^{\dagger}(\psi_1), \dots, U_c^{\dagger}(\psi_{k_1})$ are linearly independent vectors which satisfy (3.42). Hence by Theorem 3, it follows that $k_1 = m - n + k$, where k is the number of linearly independent solutions of Π_m. Therefore, applying Green's formula (2.26) to the functions y and ψ_j yields

$$\int_{\alpha}^{\beta} \overline{\psi_j(s)} f(s)\,ds + \sum_{i=1}^{p} \widehat{\psi_j}^*\big(\theta_i^+\big) S\big(\theta_i^+\big) a_i = \big(U_c^{\dagger}(\psi_j)\big)^* U(y). \tag{3.44}$$

As a result, conditions (3.39) and (3.44) show that

$$\big(U_c^{\dagger}(\psi_j)\big)^*\big(\gamma - U(y)\big) = 0 \tag{3.45}$$

for every $j = 1, \dots, k_1$. This implies the existence of ξ, and proves the sufficiency. Hence, the proof is completed. □

The particular case $m = n$ is important in many applications. For instance, if $m = n$ then Theorem 4 implies that problems Π_n and $\Pi_n^{\dagger}$ have the same number of linearly independent solutions. Moreover, if Π_n has only the trivial solution, then it follows from Theorem 5 that the solution of the inhomogeneous problem (3.37) is unique. Consequently, we proved the following corollary of Theorem 5.

Corollary 6. *If $m = n$ and the only solution of Π_n is the trivial one, then the inhomogeneous boundary value problem* (3.37) *has a unique solution.*

4. Green's functions

Suppose the rank m of the boundary form U is equal to the order n of the differential operator $\mathcal{L}$. Also suppose that problem Π_n has only the trivial solution. Now, consider the following boundary value problem:

$$\begin{cases} \mathcal{L}(x) = f(t), & t \neq \theta_i, \\ \mathcal{J}_i(x) = a_i, & i = 1, \ldots, p, \\ U(x) = 0, \end{cases} \tag{4.46}$$

and let $\Phi = [\phi_1, \ldots, \phi_n]$ be any row vector of fundamental solutions of (3.34). Then making use of the boundary formula (2.29) it is possible to express the unique solution $x = x(t)$ of (4.46) in the form

$$x(t) = \int_\alpha^\beta G(t,s) f(s)\,ds + \sum_{j=1}^p H\bigl(t, \theta_j^+\bigr) a_j, \tag{4.47}$$

where the functions $G(t,s)$ and $H(t, \theta_j^+)$ are uniquely defined by

$$G(t,s) = \begin{cases} \Phi(t)(E+K)\hat{\Phi}^{-1}(s)\frac{1}{p_0(s)} e_n, & s < t, \\ \Phi(t) K \hat{\Phi}^{-1}(s) \frac{1}{p_0(s)} e_n, & s \geqslant t, \end{cases} \tag{4.48}$$

and

$$H\bigl(t, \theta_j^+\bigr) = \begin{cases} \Phi(t)(E+K)\hat{\Phi}^{-1}(\theta_j^+), & \theta_j < t, \\ \Phi(t) K \hat{\Phi}^{-1}(\theta_j^+), & \theta_j \geqslant t, \end{cases} \tag{4.49}$$

for all $t \in J$ and $j = 1, \ldots, p$. Here the matrix K is given by

$$K = -\bigl\{M\hat{\Phi}(\alpha) + N\hat{\Phi}(\beta)\bigr\}^{-1} N\hat{\Phi}(\beta). \tag{4.50}$$

It is of great importance to note that we have a sequence $H = \langle H(t, \theta_j^+)\rangle_{j=1}^p$ of functions $H(t, \theta_j^+)$, each of which is a vector valued ($1 \times n$ matrix) function defined on J. On the other hand, the function $G : J^2 \to \mathbb{F}$ is scalar valued. We will call the pair $\{G, H\}$ the *Green's function*. Meanwhile, we remark that it is possible to denote $H_j(t) = H(t, \theta_j^+)$, but we prefer the conventional notation used for the case $n = 1$ (see [23, p. 153]). Moreover, it can be shown that these functions defined by (4.48) and (4.49) are independent of the choice of fundamental solutions $\phi_1, \ldots, \phi_n$.

In order to investigate some of the basic properties of the functions $G(t,s)$ and $H(t, \theta_j^+)$, we consider the following rectangles:

$$\begin{aligned} &R_{11} = [\alpha, \theta_1] \times [\alpha, \theta_1], \\ &R_{i1} = (\theta_{i-1}, \theta_i] \times [\alpha, \theta_1], \quad i = 2, \ldots, p+1, \\ &R_{1j} = [\alpha, \theta_1] \times (\theta_{j-1}, \theta_j], \quad j = 2, \ldots, p+1, \\ &R_{ij} = (\theta_{i-1}, \theta_i] \times (\theta_{j-1}, \theta_j], \quad i, j = 2, \ldots, p+1, \end{aligned} \tag{4.51}$$

and the triangles

$$T_{ii}^u = \bigl\{(t,s) \in R_{ii} \colon s > t\bigr\}, \qquad T_{ii}^l = \bigl\{(t,s) \in R_{ii} \colon s < t\bigr\}, \tag{4.52}$$

for all $i = 1, \dots, p+1$. In the following proposition we present the main characteristics of the function $G(t,s)$.

Proposition 7. *Let $G(t,s)$ be defined by* (4.48). *Then, the following properties hold*:

(G1) $\frac{\partial^\nu}{\partial t^\nu} G(t,s)$, $\nu = 0, 1, \dots, n-2$, *are continuous and bounded on the rectangles R_{ij},*
(G2) $\frac{\partial^\nu}{\partial t^\nu} G(t,s)$, $\nu = n-1, n$, *are continuous and bounded on the rectangles R_{ij} with $i \neq j$ and on the triangles T^u_{ii} and T^l_{ii},*
(G3) *$G(t,s)$ satisfies the jump conditions*

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G\big(s^+, s\big) - \frac{\partial^{n-1}}{\partial t^{n-1}} G\big(s^-, s\big) = \frac{1}{p_0(s)}, \quad s \neq \theta_j, \tag{4.53}$$

and

$$\hat{G}\big(\theta_j^+, \theta_j\big) - (E + B_j)\hat{G}\big(\theta_j^-, \theta_j\big) = (E + B_j)\frac{1}{p_0(\theta_j)} e_n, \tag{4.54}$$

(G4) *$G(t,s)$, considered as a function of t, is left continuous and satisfies*

$$\begin{cases} \mathcal{L}(x) = 0, & t \in J_s \setminus \langle \theta_i \rangle_{i=1}^p, \\ \mathcal{J}_i(x) = 0, & i \in \{i\colon \theta_i \in J_s\}, \\ U(x) = 0, \end{cases} \tag{4.55}$$

where J_s is any of the intervals $[\alpha, s)$ or $(s, \beta]$.

Moreover, if Π_n has only the trivial solution then the properties (G1)–(G4) *uniquely determine the function $G(t,s)$.*

The proof of the proposition is similar to that of [20, Theorem 1, p. 29] when the discontinuity is absent. However, it is of great importance to obtain the following identity:

$$\begin{aligned} \Delta \hat{G}(t, \theta_j)|_{t=\theta_j} &= \hat{G}\big(\theta_j^+, \theta_j\big) - \hat{G}\big(\theta_j^-, \theta_j\big) \\ &= (E + B_j)\frac{1}{p_0(\theta_j)} e_n + B_j \hat{G}\big(\theta_j^-, \theta_j\big), \end{aligned} \tag{4.56}$$

which directly follows from (4.48), and proves (4.54).

As it might have already been noticed, Proposition 7 does not provide an explicit form for solutions of problem (4.46). However, if Π_n has only the trivial solution then one may write the unique solution, $x = x(t)$, of the inhomogeneous problem of the form

$$\begin{cases} \mathcal{L}(x) = f(t), & t \neq \theta_i, \\ \mathcal{J}_i(x) = 0, & i = 1, \dots, p, \\ U(x) = 0, \end{cases} \tag{4.57}$$

as

$$x(t) = \int_\alpha^\beta G(t,s) f(s)\, ds. \tag{4.58}$$

Thus, we have the following theorem.

Theorem 8. *If Π_n has only the trivial solution then there exists a unique solution $x(t)$ of problem* (4.57)*, defined by* (4.58).

Proof. Consider the interiors of the rectangles R_{ij}. Then, by Proposition 7 we have

$$\frac{\partial^\nu}{\partial t^\nu}\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds=\int_{\theta_{k-1}}^{\theta_k}\frac{\partial^\nu}{\partial t^\nu}G(t,s)f(s)\,ds,\qquad \nu=0,1,\dots,n-1, \tag{4.59}$$

for every $k=1,\dots,p+1$. However, for $\nu=n$ we have the following equality:

$$\frac{\partial^n}{\partial t^n}\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds=\int_{\theta_{k-1}}^{\theta_k}\frac{\partial^n}{\partial t^n}G(t,s)f(s)\,ds+\chi_k(t)\frac{f(t)}{p_0(t)}, \tag{4.60}$$

where

$$\chi_k(t)=\begin{cases}1, & t\in(\theta_{k-1},\theta_k),\\ 0, & \text{otherwise}.\end{cases} \tag{4.61}$$

Therefore, writing (4.58) in the form

$$x(t)=\int_\alpha^\beta G(t,s)f(s)\,ds=\sum_{k=1}^{p+1}\int_{\theta_{k-1}}^{\theta_k} G(t,s)f(s)\,ds, \tag{4.62}$$

and using Eqs. (4.59) and (4.60) in (4.57) completes the proof. □

Since Theorem 8 implicitly shows that the function $G(t,s)$ is not sufficient to represent the solution of the inhomogeneous impulsive boundary value problem (4.46), we need to use the functions

$$H\big(t,\theta_j^+\big)=\big[H_1\big(t,\theta_j^+\big),\dots,H_n\big(t,\theta_j^+\big)\big],\qquad j=1,\dots,p, \tag{4.63}$$

of the sequence H. The following proposition characterizes these functions.

Proposition 9. *Let $1\leqslant j\leqslant p$ be arbitrarily fixed, and let $H(t,\theta_j^+)$ be defined by* (4.49) *with entries $H_k(t,\theta_j^+)$ as in* (4.63)*. Then, each $H_k(t,\theta_j^+)$ for $1\leqslant k\leqslant n$ is in $\mathcal{PLC}^n$ and satisfies the boundary value problem:*

$$\begin{cases}\mathcal{L}(x)=0, & t\neq\theta_i,\\ \mathcal{J}_i(x)=0, & i=1,\dots,j-1,j+1,\dots,p,\\ \mathcal{J}_j(x)=e_k, & \\ U(x)=0, & \end{cases} \tag{4.64}$$

where e_k is the kth unit vector.

Moreover, if Π_n has only the trivial solution then (4.64) *uniquely determines the functions $H(t,\theta_j^+)$ for every $j=1,\dots,p$.*

Using (4.49) and (4.63) it follows that for $t=\theta_j$ we have

$$\begin{aligned}\hat{H}\big(\theta_j^+,\theta_j^+\big)-(E+B_j)\hat{H}\big(\theta_j^-,\theta_j^+\big)&=(E+B_j)\hat{\Phi}\big(\theta_j^-\big)\hat{\Phi}\big(\theta_j^+\big)\\&=\hat{\Phi}\big(\theta_j^+\big)\hat{\Phi}^{-1}\big(\theta_j^+\big)\\&=E.\end{aligned}\tag{4.65}$$

This proves the third equation in (4.64). The remaining equations can easily be verified in a similar way.

It is not difficult, by the help of (4.65) and Proposition 9, to observe that the function $x(t)$, which is uniquely defined by

$$x(t)=\sum_{j=1}^{p}H\big(t,\theta_j^+\big)a_j,\tag{4.66}$$

satisfies only an inhomogeneous problem of the following form:

$$\begin{cases}\mathcal{L}(x)=0, & t\neq\theta_i,\\ \mathcal{J}_i(x)=a_i, & i=1,\dots,p,\\ U(x)=0.\end{cases}\tag{4.67}$$

Hence we have the following theorem.

Theorem 10. *If Π_n has only the trivial solution then there exists a unique solution $x(t)$ of problem* (4.67)*, defined by* (4.66).

Proof. Clearly, each component $H_k(t,\theta_j^+)$ of the function $H(t,\theta_j^+)$ satisfies $\mathcal{L}(x)=0$ and $U(x)=0$ for all $1\leqslant j\leqslant p$ by Proposition 9. Furthermore, it follows from (4.65) that

$$\mathcal{J}_i(x)=\mathcal{J}_i\left(\sum_{j=1}^{p}H\big(t,\theta_j^+\big)a_j\right)=a_i\tag{4.68}$$

holds for every $i=1,\dots,p$, and hence, the proof is completed. □

Consequently, combining Theorem 8 with Theorem 10 we state the following theorem, concerning the solutions of the inhomogeneous boundary value problem (4.46).

Theorem 11. *If Π_n has only the trivial solution then the solution $x=x(t)$ of* (4.46) *exists and is unique. Moreover, this solution is expressed by*

$$x(t)=\int_{\alpha}^{\beta}G(t,s)f(s)\,ds+\sum_{j=1}^{p}H\big(t,\theta_j^+\big)a_j,\tag{4.69}$$

where $\{G(t,s),\langle H(t,\theta_j^+)\rangle_{j=1}^{p}\}$ is the Green's function.

The proof directly follows from Theorems 8 and 10 by making use of the properties of the Green's function characterized by Propositions 7 and 9.

We conclude this section by giving an example of a Green's function for a specific second order boundary value problem.

4.1. An example

Consider the following inhomogeneous boundary value problem:

$$\begin{cases} -x'' = 2, \quad t \neq 1, \\ \Delta \hat{x}|_{t=1} - \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \hat{x}(1^-) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \\ x(0) = x(\pi) = 0, \end{cases} \tag{4.70}$$

on the interval $J = [0, \pi]$. When the associated homogeneous problem is considered it can easily be shown that the function $G(t,s)$ is expressed by

$$G(t,s) = \begin{cases} s + \frac{\pi s}{2-3\pi} t, & 0 \leqslant s < t \leqslant 1, \\ t + \frac{\pi t}{2-3\pi} s, & 0 \leqslant t \leqslant s \leqslant 1, \\ -\frac{1}{2} \frac{\pi(2-3s)}{2-3\pi} t + \frac{1}{2} st, & 0 \leqslant t \leqslant 1 < s \leqslant \pi, \\ -\frac{\pi(2-3t)}{2-3\pi} s + ts, & 0 \leqslant s \leqslant 1 < t \leqslant \pi, \\ -\frac{1}{2}\left[(2-3s)t + \frac{\pi(2-3s)}{2-3\pi}(-2+3t)\right], & 1 < s < t \leqslant \pi, \\ -\frac{1}{2}\left[(2-3t)s + \frac{\pi(2-3t)}{2-3\pi}(-2+3s)\right], & 1 < t \leqslant s \leqslant \pi, \end{cases} \tag{4.71}$$

while the function $H(t, 1^+)$ is defined by

$$H(t, 1^+) = \begin{cases} \frac{1}{2}\left[\frac{2}{2-3\pi} t, \ \frac{-2+2\pi}{2-3\pi}\right], & 0 \leqslant t \leqslant 1, \\ \frac{1}{2}\left[3t + \frac{3\pi}{2-3\pi}(-2+3t), \ -t - \frac{\pi}{2-3\pi}(-2+3t)\right], & 1 \leqslant t \leqslant \pi. \end{cases} \tag{4.72}$$

Note that this is the only function of the sequence H, for only at $t = 1$ does the problem (4.70) have a discontinuity. Therefore, the unique solution $x = x(t)$ of the inhomogeneous prob-

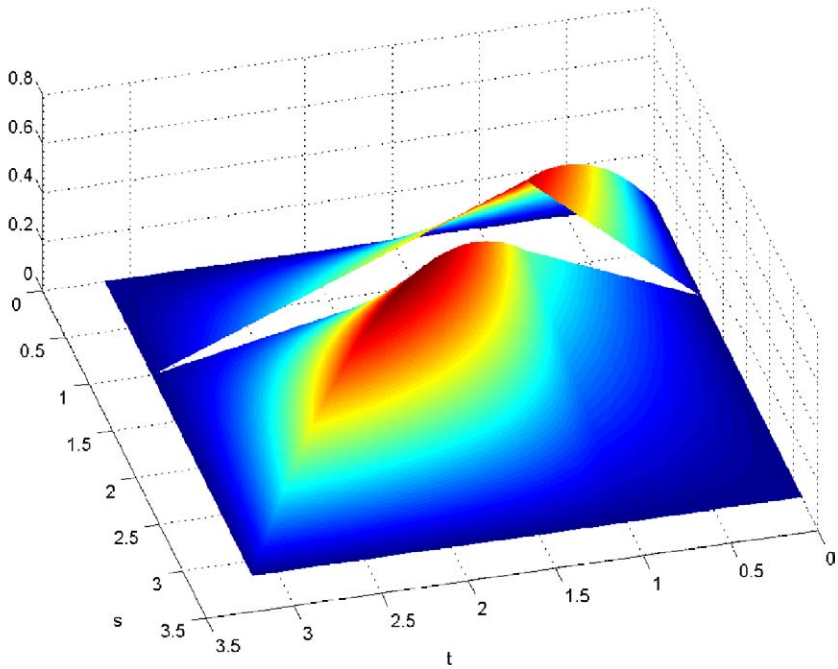

Fig. 1. The graph of the function $G(t,s)$.

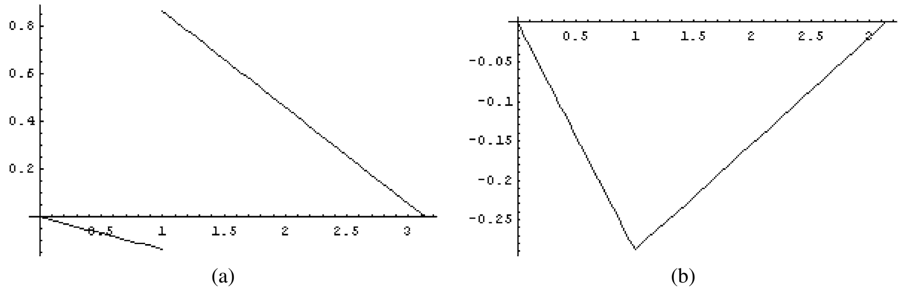

Fig. 2. The graphs of the components of the function $H(t,1^+)$: (a) graph of $H_1(t,1^+)$, (b) graph of $H_2(t,1^+)$.

lem (4.70) can be calculated as

$$x(t)=\int_0^\pi 2G(t,s)\,ds+H\big(t,1^+\big)\begin{pmatrix}0\\-4\end{pmatrix}$$
$$=\begin{cases}\frac{\pi^2+7\pi-7}{3\pi-2}t-t^2, & 0\leqslant t\leqslant 1,\\ \frac{(7-2\pi)\pi}{3\pi-2}+\frac{3\pi^2-7}{3\pi-2}t-t^2, & 1<t\leqslant\pi.\end{cases} \tag{4.73}$$

In order to visualize the Green's function, in Fig. 1 the graph of the function $G(t,s)$ is shown, while in Fig. 2 we present the graphs of the components $H_1(t,1^+)$ and $H_2(t,1^+)$ of the vector valued function $H(t,1^+)$.

5. Eigenvalue problems

In many applications of boundary value problems one needs to deal with the following eigenvalue problem

$$\begin{cases}\mathcal{L}(x)=\lambda x, & t\neq\theta_i,\\ \mathcal{J}_i(x)=0, & i=1,\dots,p,\\ U(x)=0,\end{cases} \tag{5.74}$$

where the boundary form U is of rank m. Consider the impulsive differential operator $\mathcal{L}_0: D_0\to\mathcal{PLC}$ defined by the differential operator $\mathcal{L}$, on the linear subspace

$$D_0=\big\{x\in\mathcal{PLC}^n\colon\ U(x)=0,\ \mathcal{J}_i(x)=0,\ i=1,\dots,p\big\} \tag{5.75}$$

of the space $\mathcal{PLC}$. That is to say, we are interested in the eigenvalue problem

$$\mathcal{L}_0x=\lambda x,\quad \text{for } x\in D_0. \tag{5.76}$$

In order to characterize the eigenvalues of the operator $\mathcal{L}_0$, we need the analytical properties of the solutions of problem (5.74) with respect to the parameter λ. Fortunately, using [3,11] it can be shown that the solution $x=x(t,\lambda)$ of the linear homogeneous impulsive equation

$$\begin{cases}\mathcal{L}(x)=\lambda x, & t\neq\theta_i,\\ \mathcal{J}_i(x)=0, & i=1,\dots,p,\end{cases} \tag{5.77}$$

is an entire function of the parameter λ for fixed $t\in J=[\alpha,\beta]$. Moreover, it is easy to show that $x^{(j)}(t,\lambda)$ for fixed $t\neq\theta_i$ and $x^{(j)}(\theta_i^{\pm},\lambda)$ are entire functions in λ for every $j=1,\dots,n-1$ and $i=1,\dots,p$.

In addition, to achieve a condition for the determination of eigenvalues of the operator $\mathcal{L}_0$ let

$$\phi_1(t,\lambda),\dots,\phi_n(t,\lambda) \tag{5.78}$$

be fundamental solutions of (5.77). To have a nontrivial solution of the eigenvalue problem (5.74) it is necessary and sufficient that there are constants $c_1,\dots,c_n$, not all zero, so that

$$x(t,\lambda)=\sum_{j=1}^{n} c_j\phi_j(t,\lambda) \tag{5.79}$$

satisfies the boundary condition $U(x)=0$. In other words, if we let $\Phi=[\phi_1,\dots,\phi_n]$ to be the row vector of fundamental solutions then the system of equations

$$(U\hat{\Phi})c=0 \tag{5.80}$$

should have a nontrivial solution $c=[c_1,\dots,c_n]^T$. Since functions in (5.78) and their derivatives with respect to $t\neq\theta_i$ up to and including the order $n-1$ are entire functions of λ for fixed $t\in J$, it follows that the matrix $U\hat{\Phi}(t,\lambda)$ is an entire function of λ. Therefore, we may state the following theorem, which has almost the same statement as in [20, p. 14] for eigenvalue problems that have no discontinuity points in the interval J.

Theorem 12. *For any impulsive differential operator $\mathcal{L}_0$ only the following two possibilities can occur*:

(1) *every number λ is an eigenvalue of $\mathcal{L}_0$, or*
(2) *the operator $\mathcal{L}_0$ has at most enumerable eigenvalues* (*in particular, none at all*), *and these eigenvalues can have no finite accumulation point.*

The case $m=n$ is of particular interest in many applications of eigenvalue problems. In the rest of this section we will assume $m=n$, unless otherwise explicitly stated. Now, in this case we may define the characteristic determinant

$$\gamma(\lambda)=\det U\hat{\Phi}(t,\lambda) \tag{5.81}$$

for the operator $\mathcal{L}_0$ and state the following corollary of Theorem 12.

Corollary 13. *Let $m=n$. The eigenvalues of the operator $\mathcal{L}_0$ are the zeros of the characteristic determinant $\gamma(\lambda)$. If $\gamma(\lambda)$ vanishes identically, then any number λ is an eigenvalue. However, if $\gamma(\lambda)$ is not identically zero, then $\mathcal{L}_0$ has at most enumerable eigenvalues, and these eigenvalues can have no finite accumulation point.*

As can be guessed, many properties of the eigenvalues of linear operators are valid for the operator $\mathcal{L}_0$ defined on D_0. For instance, if $\mathcal{L}_0^\dagger$, the adjoint operator for $\mathcal{L}_0$, is defined by $\mathcal{L}^\dagger$ for $t\neq\theta_i$ on the space of functions

$$D_0^\dagger=\{y\in\mathcal{PLC}^n\colon\ U^\dagger(y)=0,\ \mathcal{J}_i^\dagger(y)=0,\ i=1,\dots,p\}, \tag{5.82}$$

then one can easily show that if λ is an eigenvalue of $\mathcal{L}_0$ with multiplicity k, then $\bar{\lambda}$ is an eigenvalue of the adjoint operator $\mathcal{L}_0^\dagger$, and has the same multiplicity k.

Moreover, if we let x to be an eigenfunction of $\mathcal{L}_0$ associated with the eigenvalue λ, and y be an eigenfunction of $\mathcal{L}_0^\dagger$ associated with the eigenvalue μ, then it follows from Green's formula (2.26) that

$$0 = (\lambda - \bar{\mu})\langle x, y\rangle \tag{5.83}$$

holds, where

$$\langle x, y\rangle = \int_\alpha^\beta \overline{y(s)}x(s)\,ds \tag{5.84}$$

is the standard inner product for $\mathcal{PLC}$. Therefore, using (5.83) one can prove the following theorem.

Theorem 14. *Let $m = n$ for the operator $\mathcal{L}_0$. Then,*

(1) *the eigenfunctions x and y of the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\dagger$ associated with the eigenvalues λ and μ, respectively, are orthogonal if $\lambda \neq \bar{\mu}$*;
(2) *the eigenvalues of a self-adjoint operator $\mathcal{L}_0$ are real*;
(3) *the eigenfunctions of a self-adjoint operator $\mathcal{L}_0$ associated to distinct eigenvalues are orthogonal.*

A particular application of Theorem 14, for instance, arises when the properties of eigensolutions of Sturm–Liouville eigenvalue problems [11,29] with impulses are investigated.

Now, consider an inhomogeneous boundary value problem:

$$\begin{cases} \mathcal{L}(x) = \lambda x + f(t), & t \neq \theta_i, \\ \mathcal{J}_i(x) = a_i, & i = 1, \dots, p, \\ U(x) = 0, \end{cases} \tag{5.85}$$

where U is assumed to have rank n, and f is in $\mathcal{PLC}$ and $a = \langle a_i\rangle_{i=1}^p$ is a given sequence of vectors a_i in $\mathbb{F}^n$. Let $\mathcal{L}_a : D_a \to \mathcal{PLC}$ be an impulsive differential operator defined by $\mathcal{L}$ for $t \neq \theta_i$ on the space of functions

$$D_a = \bigl\{x \in \mathcal{PLC}^n \colon\ U(x) = 0,\ \mathcal{J}_i(x) = a_i,\ i = 1, \dots, p\bigr\}. \tag{5.86}$$

Then it is possible to rewrite (5.85) in the form

$$\mathcal{L}_a x = \lambda x + f(t), \tag{5.87}$$

for functions x in D_a. The eigenvalues of the operator $\mathcal{L}_0$ strongly affect the existence of solutions of (5.87). In fact, the following theorem is a consequence of the preceding Theorem 5.

Theorem 15. *The problem* (5.87) *has a solution if and only if the equality*

$$\int_\alpha^\beta \overline{\psi(s)} f(s)\,ds + \sum_{i=1}^p \hat{\psi}^*\bigl(\theta_i^+\bigr)S\bigl(\theta_i^+\bigr)a_i = 0 \tag{5.88}$$

holds for every solution ψ of the adjoint problem $\mathcal{L}_0^\dagger y = \bar{\lambda}y$, for $y \in D_0^\dagger$.

In particular, since (5.87) is an inhomogeneous boundary value problem containing a parameter λ, it is possible to write its solution in terms of the Green's function. To achieve this, suppose λ is not an eigenvalue of the operator $\mathcal{L}_0$. This means $\mathcal{L}_0 x = \lambda x$ has only the trivial solution, and hence, by the results obtained in Section 4 it follows that there exists a unique Green's function $\{G(t,s,\lambda), \langle H(t,\theta_i^+,\lambda)\rangle_{i=1}^p\}$, which depends on λ. As a result, the solution $x = x(t,\lambda)$ of (5.87) is given by

$$x(t,\lambda) = \int_\alpha^\beta G(t,s,\lambda) f(s)\,ds + \sum_{i=1}^p H\big(t,\theta_i^+,\lambda\big)a_i. \tag{5.89}$$

This proves the following theorem.

Theorem 16. *If λ is not an eigenvalue of $\mathcal{L}_0$, then for any f in $\mathcal{PLC}$ and any $a = \langle a_i\rangle_{i=1}^p$ with $a_i \in \mathbb{F}^n$, the problem* (5.87) *has a unique solution $x = x(t,\lambda)$ defined by* (5.89) *in terms of the Green's function.*

This theorem is important for further study of eigenvalue problems. For instance, if $\lambda = 0$ is not an eigenvalue of $\mathcal{L}_0$, then Eq. (5.87) can be written formally as an integral equation of Fredholm type:

$$x(t) = \lambda \int_\alpha^\beta G(t,s) f(s)\,ds + g(t), \tag{5.90}$$

where $G(t,s) = G(t,s,0)$ and

$$g(t) = \int_\alpha^\beta G(t,s) f(s)\,ds + \sum_{i=1}^p H\big(t,\theta_i^+\big)a_i,$$

with $H(t,\theta_i^+) = H(t,\theta_i^+,0)$.

It is of particular interest to investigate integral equations [16,24,25] of the form (5.90), where $G(t,s)$ is a piecewise defined kernel and $g(t)$ is a piecewise defined function, having a certain set of properties.

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