

# Asymptotic equivalence of differential equations and asymptotically almost periodic solutions

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## Abstract

In this paper we use Rab's lemma [M. Ráb, Über lineare perturbationen eines systems von linearen differentialgleichungen, Czechoslovak Math. J. 83 (1958) 222–229; M. Ráb, Note sur les formules asymptotiques pour les solutions d'un système d'équations différentielles linéaires, Czechoslovak Math. J. 91 (1966) 127–129] to obtain new sufficient conditions for the asymptotic equivalence of linear and quasilinear systems of ordinary differential equations. Yakubovich's result [V.V. Nemytskii, V.V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, New Jersey, 1966; V.A. Yakubovich, On the asymptotic behavior of systems of differential equations, Mat. Sb. 28 (1951) 217–240] on the asymptotic equivalence of a linear and a quasilinear system is developed. On the basis of the equivalence, the existence of asymptotically almost periodic solutions of the systems is investigated. The definitions of biasymptotic equivalence for the equations and biasymptotically almost periodic solutions are introduced. Theorems on the sufficient conditions for the systems to be biasymptotically equivalent and for the existence of biasymptotically almost periodic solutions are obtained. Appropriate examples are constructed.

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## 1. Introduction and preliminaries

Let  $\mathbb{N}$  and  $\mathbb{R}$  be sets of all natural and real numbers, respectively. Denote by  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , and by  $C(\mathbb{X}, \mathbb{Y})$  the space of all continuous functions defined on  $\mathbb{X}$  with values in  $\mathbb{Y}$ .

We shall recall the definitions of almost periodic and asymptotically almost periodic functions; see [1,5,7] for more details.

A number  $\tau \in \mathbb{R}$  is called an  $\epsilon$ -translation number of a function  $f \in C(\mathbb{R}, \mathbb{R}^n)$  if  $\|f(t + \tau) - f(t)\| < \epsilon$  for all  $t \in \mathbb{R}$ . A function  $f \in C(\mathbb{R}, \mathbb{R}^n)$  is called almost periodic if for a given  $\epsilon \in \mathbb{R}$ ,  $\epsilon > 0$ , there exists a relatively dense set of  $\epsilon$ -translation numbers of  $f$ . The set of all almost periodic functions is denoted by  $\mathcal{AP}(\mathbb{R})$ .

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A function  $f \in C(\mathbb{R}, \mathbb{R}^n)$  is called asymptotically almost periodic if there is a function  $g \in \mathcal{AP}(\mathbb{R})$  and a function  $\phi \in C(\mathbb{R}, \mathbb{R}^n)$  with  $\lim_{t \rightarrow \infty} \phi(t) = 0$  such that  $f(t) = g(t) + \phi(t)$ .

The basic definition of an almost periodic function given by Bohr has been modified by several authors [1,5,7,11, 21,27]. Below we introduce a new notion with regard to almost periodic functions.

**Definition 1.1.** A function  $f \in C(\mathbb{R}, \mathbb{R}^n)$  is called biasymptotically almost periodic if  $f(t) = g(t) + \phi(t)$  for some  $g \in \mathcal{AP}(\mathbb{R})$  and  $\phi \in C(\mathbb{R}, \mathbb{R}^n)$  with  $\lim_{t \rightarrow \pm\infty} \phi(t) = 0$ .

Note that every biasymptotically almost periodic function is a pseudo-almost periodic function [27], but not conversely.

In this paper we are concerned with the linear system

$$y' = [A(t) + B(t)]y, \tag{1}$$

which may be viewed as a perturbation of

$$x' = A(t)x, \tag{2}$$

where  $x, y \in \mathbb{R}^n$ , and  $A, B \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ .

Moreover we consider the quasilinear systems of the form

$$y' = Cy + f(t, y) \tag{3}$$

and the corresponding homogeneous linear system

$$x' = Cx, \tag{4}$$

where  $x, y \in \mathbb{R}^n$ ,  $C \in \mathbb{R}^{n \times n}$ , and  $f(t, x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$  such that

$$f(t, 0) = 0 \quad \text{for all } t \in \mathbb{R}.$$

**Definition 1.2** ([2,16]). A homeomorphism  $\mathcal{H}$  between the sets of solutions  $x(t)$  and  $y(t)$  is called an asymptotic equivalence if  $y(t) = \mathcal{H}(x(t))$  implies that  $x(t) - y(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Definition 1.3.** A homeomorphism  $\mathcal{H}$  between the sets of solutions  $x(t)$  and  $y(t)$  is called a biasymptotic equivalence if  $y(t) = \mathcal{H}(x(t))$  implies that  $x(t) - y(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$ .

Our main objective is to investigate the problem of asymptotic equivalence of systems and to prove the existence of asymptotically and biasymptotically almost periodic solutions of (1) and (3).

The classical theorem of Levinson [14] states that if the trivial solution of (2) is uniformly stable,  $A(t) \equiv A$ , and

$$\int_0^\infty \|B(t)\| dt < \infty, \tag{5}$$

then (1) and (2) are asymptotically equivalent. In the case when  $A$  is not a constant matrix, Wintner [28] proved that the above conclusion remains valid if all solutions of (2) are bounded, (5) is satisfied, and

$$\liminf_{t \rightarrow \infty} \int_0^t \text{Trace}[B(s)] ds > -\infty.$$

Later, Yakubovich [25] considered (3) and obtained asymptotic equivalence of (3) and (4); see [16,25] for details. After the pioneering works of Levinson, Wintner, and Yakubovich, the problem of asymptotic equivalence of differential systems including linear, nonlinear, and functional equations was investigated by many authors; see e.g. [2–4,6,10, 14–20,22–27], and the references cited therein. Two interesting articles in this direction which also motivate our study here in this paper were written by Ráb [18,19]. In fact, the main result in [19] is an improvement of the earlier one in [18], which we have employed in our work.

Asymptotically almost periodic functions were introduced by Fréchet [8,9]. The existence of this type of solutions was investigated by Fink [7] (Theorem 9.5) for the first time. For more results on the existence of asymptotically almost

periodic solutions of different types of equations we refer the reader to [12,13,15,17,24,26] and the references cited therein. In this work we exploit the idea of Fink to obtain the existence of asymptotically almost periodic solutions of linear and quasilinear systems. Moreover, we prove a theorem about biasymptotic equivalence of linear systems and a theorem on the existence of biasymptotically almost periodic solutions. Apparently the notions of biasymptotic equivalence and a biasymptotically almost periodic function are introduced for the first time in this paper.

The paper is organized as follows. In the next section, we prove a main lemma of Ráb and obtain sufficient conditions concerning the asymptotic equivalence of (1) and (2), and the existence of a family of asymptotically almost periodic solutions of the system (1). The third section is devoted to the problem of the asymptotic equivalence of systems (4) and (3) and the problem of existence of asymptotically almost periodic solutions of the system (3). The last section concerns with biasymptotic equivalence problem and the existence of biasymptotically almost periodic solutions of (1). In addition, examples are given to illustrate the results.

## 2. Asymptotic equivalence of linear systems and asymptotically almost periodic solutions

Let  $X(t)$ ,  $X(0) = I$ , be a fundamental matrix solution of (2). Setting  $y = X(t)u$ , we easily see from (1) that

$$u' = P(t)u, \quad (6)$$

where  $P(t) = X^{-1}(t)B(t)X(t)$ .

Assume that

$$(C_1) \int_0^\infty \|P(t)\| dt < \infty.$$

The following lemma has been obtained Ráb [18,19], for which we include a proof for convenience.

**Lemma 2.1.** *If (C<sub>1</sub>) is valid then the matrix differential equation*

$$\Psi' = P(t)(\Psi + I) \quad (7)$$

*has a solution  $\Psi(t)$  which satisfies  $\Psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

**Proof.** Construct a sequence of  $n \times n$  matrices  $\{\Psi_k\}$  defined on  $R_+ = [0, \infty)$  as follows:

$$\Psi_0(t) = I, \quad \Psi_k(t) = - \int_t^\infty P(s) \Psi_{k-1}(s) ds \quad \text{for } k = 1, 2, \dots$$

Fix  $\epsilon$ ,  $0 < \epsilon < 1$ . In view of (C<sub>1</sub>) there exists a  $t_1 > 0$  such that

$$\int_t^\infty \|P(s)\| ds < \epsilon \quad \text{for all } t > t_1.$$

It follows that  $\|\Psi_k(t)\| < \epsilon^k$ ,  $k \in N$ , and consequently the series  $\sum_{k=1}^\infty \Psi_k(t)$  is convergent uniformly for  $t \in [t_1, \infty)$ . Letting  $\Psi(t) = \sum_{k=1}^\infty \Psi_k(t)$ , one can easily check that  $\Psi$  satisfies

$$\Psi(t) = - \int_t^\infty P(s)[I + \Psi(s)] ds \quad (8)$$

and hence it is a solution (7). From (8) it also follows that  $\Psi \rightarrow 0$  as  $t \rightarrow \infty$ , which completes that proof.  $\square$

We may assume that

$$(C_2) \lim_{t \rightarrow \infty} X(t) \Psi(t) = 0.$$

**Theorem 2.1.** *Suppose that conditions (C<sub>1</sub>) and (C<sub>2</sub>) hold. Then (1) and (2) are asymptotically equivalent.*

**Proof.** Let  $t$  be sufficiently large,  $t \geq t_1$  say. In view of (8) we see that the function  $u(t) = [I + \Psi(t)]c$ ,  $c \in \mathbb{R}^n$ , is a solution of (6) defined on  $[t_1, \infty)$  and hence

$$y(t) = X(t)[I + \Psi(t)]c \quad (9)$$

is a solution of (1).

Since  $\Psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ , there exists a  $t_2 > t_1$  such that  $I + \Psi(t_2)$  is nonsingular. Let  $x^0 = X(t_2)c$  and  $y^0 = X(t_2)(I + \Psi(t_2))c$ . Denote by  $y(t, c) = y(t, t_2, x^0)$  and  $x(t, c) = x(t, t_2, y^0)$  the solutions of (1) and (2) satisfying  $x(t_2) = x^0$  and  $y(t_2) = y^0$ , respectively. Now, because of the existence and uniqueness of solutions of linear differential equations and the fact that  $I + \Psi(t_2)$  is nonsingular, the relation

$$y^0 = X(t_2)[I + \Psi(t_2)]X^{-1}(t_2)x^0$$

defines an isomorphism between solutions  $x(t)$  of (2) and  $y(t)$  of (1) such that

$$y(t) = x(t) + X(t)\Psi(t)c$$

for  $t > t_1$ . The last equality, in view of (C<sub>2</sub>), completes the proof.  $\square$

**Corollary 2.1.** *Suppose that the system (2) has a  $k$ -parameter ( $k \leq n$ ) family  $\sigma_2$  of almost periodic solutions, and that the conditions (C<sub>1</sub>), (C<sub>2</sub>) are satisfied. Then there exists a  $k$ -parameter family  $\sigma_1$  of asymptotic almost periodic solutions of (1), and  $\sigma_1$  is isomorphic to  $\sigma_2$ .*

**Example 2.1.** Consider the systems

$$x'' - 2(t + 1)^{-2}x = 0 \tag{10}$$

and

$$y'' - [2(t + 1)^{-2} + b(t)]y = 0, \tag{11}$$

where  $b(t)$  is a continuous function defined on  $R_+$ . We assume that there exist real numbers  $K_1 > 0$  and  $\alpha > 0$  such that

$$|b(t)| < K_1e^{-\alpha t} \quad \text{for all } t \in R_+. \tag{12}$$

Notice that (10) has solutions  $x_1(t) = (t + 1)^2$  and  $x_2(t) = (t + 1)^{-1}$ .

If we transform the above second-order equations into systems of the form (1) and (2) we identify that

$$A = \begin{bmatrix} 0 & 1 \\ 2/(t + 1)^2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ b(t) & 0 \end{bmatrix}.$$

It is easy to see that for a given  $\epsilon > 0$  there exists  $K > 0$  such that

$$\|P(t)\| \leq Ke^{(-\alpha + \epsilon)t} \quad \text{for all } t \in R_+. \tag{13}$$

Fix  $\epsilon$  so that  $\beta := \alpha + \epsilon < 0$ . Then (6) is satisfied, i.e.,

$$\int_0^\infty \|P(t)\|dt < \infty,$$

and

$$\|\Psi(t)\| \leq e^{K|\beta|^{-1}e^{\beta t}} - 1. \tag{14}$$

Moreover, using (13) and (14) one can show that  $X(t)\Psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ , i.e., (C<sub>2</sub>) holds.

Since the conditions of Theorem 2.1 are fulfilled, we may conclude that (10) and (11) are asymptotically equivalent whenever (12) holds.

**Example 2.2.** Let  $b(t)$  be a continuous function such that  $|b(t)| \leq K_1e^{-\alpha t}$  for all  $t \in R_+$  for some  $\alpha > 0$ ,  $K_1 > 0$ , and  $C \in \mathbb{R}^{5 \times 5}$ . Consider

$$y' = (A + B(t))y, \tag{15}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi & 0 \\ 0 & 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix},$$

$B(t) = b(t)C$ , and  $\beta > 0$  satisfies  $\alpha - 2\beta > 0$ .

The associated equation  $x' = Ax$  has a fundamental matrix

$$X(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0 & 0 \\ -\sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \cos \pi t & \sin \pi t & 0 \\ 0 & 0 & \sin \pi t & \cos \pi t & 0 \\ 0 & 0 & 0 & 0 & e^{\beta t} \end{pmatrix}.$$

The equality

$$P(t) = X^{-1}(t)B(t)X(t) = b(t)X^{-1}(t)CX(t)$$

implies that there exists a  $K > 0$  such that  $\|P(t)\| \leq Ke^{-(\alpha-\beta)t}$  for all  $t \in R_+$ . Therefore, (C<sub>1</sub>) is valid. We also see that

$$\|\Psi(t)\| \leq \sum_{k=1}^{\infty} \frac{(Ke^{-(\alpha-\beta)t})^k}{(\alpha - \beta)^k k!} = 1 - e^{K(\alpha-\beta)^{-1}e^{-(\alpha-\beta)t}},$$

and hence  $X(t)\Psi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In view of Corollary 2.1 we conclude that system (15) has a 4-parameter family of asymptotically almost periodic solutions. More precisely they are asymptotically “quasiperiodic” solutions and every such solution has a torus as the  $\omega$ -limit set.

### 3. Asymptotic equivalence of linear and quasilinear systems

Let  $\alpha = \min_j \Re\lambda_j$  and  $\beta = \max_j \Re\lambda_j$ , where  $\Re\lambda_j$  denotes the real part of the eigenvalue  $\lambda_j$  of the matrix  $C$ . Let  $m_\alpha$  and  $m_\beta$  be the maxima of degrees of elementary divisors of  $C$  corresponding to eigenvalues with real part equal to  $\alpha$  and  $\beta$ , respectively. Clearly, there exist constants  $\kappa_1, \kappa_2$  such that

$$\|e^{Ct}\| \leq \kappa_1 t^{m_\beta-1} e^{\beta t} \quad \text{and} \quad \|e^{-Ct}\| \leq \kappa_2 t^{m_\alpha-1} e^{-\alpha t}$$

for all  $t \in R_+ = [0, \infty)$ .

The following conditions are to be assumed:

- (C<sub>3</sub>)  $\|f(t, x_1) - f(t, x_2)\| \leq \eta(t)\|x_1 - x_2\|$  for all  $(t, x_1), (t, x_2) \in R_+ \times R^n$ , and for some nonnegative function  $\eta(t)$  defined on  $R_+$ ;
- (C<sub>4</sub>)  $L := \int_0^\infty t^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)t} \eta(t) dt < \infty$ .

**Lemma 3.1.** *If (C<sub>3</sub>) and (C<sub>4</sub>) are valid, then every solution of*

$$u' = e^{-Ct} f(t, e^{Ct} u) \tag{16}$$

is bounded on  $R_+$  and for each solution  $u$  of (16) there exists a constant vector  $c_u \in \mathbb{R}^n$  such that  $u(t) \rightarrow c_u$  as  $t \rightarrow \infty$ .

**Proof.** Let  $u(t) = u(t, t_0, u_0)$  denote the solution of (16) satisfying  $u(t_0) = u_0, t_0 \geq 0$ . It is clear that

$$u(t) = u_0 + \int_{t_0}^t e^{-Cs} f(s, e^{Cs} u(s)) ds, \quad t \geq t_0.$$

By using (C<sub>3</sub>) and  $f(t, 0) = 0$ , we see that

$$|u(t)| \leq |u_0| + k_1 \int_{t_0}^t s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) |u(s)| ds, \quad t \geq t_0$$

for some  $k_1 > 0$ . In view of (C<sub>4</sub>) and Gronwall’s inequality, we have

$$|u(t)| \leq |u_0| e^{k_1 \int_{t_0}^t s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) ds} \leq |u_0| e^{k_1 L} < \infty, \quad t \geq t_0.$$

Let  $M_0 = \max\{|u(t)| : t \in [0, t_0]\}$  and  $M = \max\{M_0, |u_0| e^{k_1 L}\}$ . Then we have  $|u(t)| \leq M$  for all  $t \in R_+$ .

To prove the second part of the theorem, we first note that

$$\left| \int_{t_0}^t e^{-Cs} f(s, e^{Cs} u(s)) ds \right| \leq Mk_1 \int_0^\infty t^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)t} \eta(t) dt < \infty.$$

So we may define

$$c_u = u_0 + \int_{t_0}^\infty e^{-Cs} f(s, e^{Cs} u(s)) ds.$$

It follows that

$$u(t) = c_u - \int_t^\infty e^{-Cs} f(s, e^{Cs} u(s)) ds,$$

which completes the proof.  $\square$

The following lemma can be easily justified by a direct substitution.

**Lemma 3.2.** *If  $y(t)$  is a solution of (3), then there is a solution  $u(t)$  of (16) such that*

$$y(t) = e^{Ct} u(t). \tag{17}$$

*Conversely, if  $u(t)$  is a solution of (16) then  $y(t)$  in (17) is a solution of (3).*

**Theorem 3.1.** *If conditions (C<sub>3</sub>) and (C<sub>4</sub>) are satisfied, then every solution  $y(t)$  of (3) possesses an asymptotic representation of the form*

$$y(t) = e^{Ct} [c + o(1)],$$

where  $c \in \mathbb{R}^n$  is a constant vector and for a solution  $u(t)$  of (16),

$$o(1) = - \int_t^\infty e^{-Cs} f(s, e^{Cs} u(s)) ds.$$

**Proof.** The proof follows from Lemmas 3.1 and 3.2.  $\square$

**Theorem 3.2.** *Assume that (C<sub>3</sub>) and (C<sub>4</sub>) are fulfilled, and*

$$(C_5) \lim_{t \rightarrow \infty} \int_t^\infty (s-t)^{m_\alpha-1} s^{m_\beta-1} e^{\alpha(t-s)} e^{\beta s} \eta(s) ds = 0.$$

*Then (3) and (4) are asymptotically equivalent.*

**Proof.** In view of Lemma 3.1 we see that

$$\begin{aligned} y(t) &= e^{Ct} \left[ c_u - \int_t^\infty e^{-Cs} f(s, e^{Cs} u(s)) ds \right] \\ &= x(t) - \int_t^\infty e^{C(t-s)} f(s, e^{Cs} u(s)) ds, \end{aligned}$$

where  $x(t) = e^{Ct} c_u$  is a solution of (4) and  $u(t) = u(t, t_0, u_0)$  is a solution of (16). It is clear that a given  $u_0$  results in a homeomorphism between  $x(t)$  and  $y(t)$ . In view of (C<sub>5</sub>), we also see that  $x(t) - y(t) \rightarrow 0$  as  $t \rightarrow \infty$ , which completes the proof of the theorem.  $\square$

In [25], Yakubovich proved that if

$$\lim_{t \rightarrow \infty} \int_t^\infty s^{m_\beta+m_\alpha-2} e^{\beta s} \eta(s) ds = 0 \tag{18}$$

then (3) and (4) are asymptotically equivalent. It is clear that if  $\alpha > 0$  then condition (C<sub>5</sub>) is weaker than (18).

The following assertion is a simple corollary of Theorem 3.2

**Corollary 3.1.** *Suppose that conditions (C<sub>3</sub>), (C<sub>4</sub>), (C<sub>5</sub>) hold, and that system (4) has a  $k$ -parameter ( $k \leq n$ ) family  $\gamma_1$  of almost periodic solutions. Then (3) admits a  $k$ -parameter family  $\gamma_2$  of asymptotically almost periodic solutions, and  $\gamma_1$  is homeomorphic  $\gamma_2$ .*

#### 4. Biasymptotic equivalence of linear systems and biasymptotically almost periodic solutions

With regard to systems (1) and (2) we shall make use of the following conditions:

(C<sub>6</sub>)  $A(-t) = -A(t)$  for all  $t \in \mathbb{R}$ .

(C<sub>7</sub>)  $B(-t) = B(t)$  for all  $t \in \mathbb{R}$ .

We will rely on the following two lemmas. The first lemma is almost trivial.

**Lemma 4.1.** *If (C<sub>6</sub>) is satisfied then  $X(-t) = X(t)$  for all  $t \in \mathbb{R}$ , and if in addition (C<sub>7</sub>) holds then  $P(-t) = -P(t)$  for all  $t \in \mathbb{R}$ .*

**Lemma 4.2.** *Assume that conditions (C<sub>1</sub>), (C<sub>6</sub>), (C<sub>7</sub>) are valid. Then (7) has a solution  $\Psi(t)$  which satisfies  $\Psi(-t) = \Psi(t)$  for all  $t \in \mathbb{R}$  and  $\Psi \rightarrow 0$  as  $t \rightarrow \infty$ .*

**Proof.** By Lemma 2.1 there exists a solution  $\Psi_+(t)$  of (7) which is defined for  $t \geq t_1$  and satisfies  $\Psi_+ \rightarrow 0$  as  $t \rightarrow \infty$ .

Using  $P(-t) = -P(t)$  we see that

$$\int_{-\infty}^{-t_1} \|P(s)\| ds = \int_{t_1}^{\infty} \|P(s)\| ds < \epsilon.$$

We may define a sequence of  $n \times n$  matrices  $\{\tilde{\Psi}_k\}$  for  $t \in (-\infty, 0]$  as follows:

$$\tilde{\Psi}_0(t) = I, \quad \tilde{\Psi}_k = \int_{-\infty}^t P(s) \tilde{\Psi}_{k-1}(s) ds \quad \text{for } k = 1, 2, \dots$$

As in the proof of Lemma 2.1, the matrix function  $\Psi_-(t) = \sum_{k=1}^{\infty} \tilde{\Psi}_k(t)$  satisfies

$$\Psi(t) = \int_{-\infty}^t P(s)(I + \Psi(s)) ds$$

and hence becomes a solution of (7) for  $t \leq -t_1$ .

On the other hand, since  $\Psi_0(-t) = \tilde{\Psi}_0(t) = I$  we have

$$\begin{aligned} \Psi_1(-t) &= - \int_{-t}^{\infty} P(s) \Psi_0(s) ds = \int_t^{-\infty} P(-s) \Psi_0(-s) ds \\ &= \int_{-\infty}^t P(s) \tilde{\Psi}_0(s) ds = \tilde{\Psi}_1(t). \end{aligned}$$

It follows by induction that  $\Psi_k(-t) = \tilde{\Psi}_k(t)$  for all  $k = 0, 1, 2, \dots$  and for all  $t \leq -t_1$ . Hence  $\Psi_+(-t) = \Psi_-(t)$  for all  $t \leq -t_1$ . Continuing  $\Psi_+$  and  $\Psi_-$  as solutions of (7), one can obtain that  $\Psi_+(-t) = \Psi_-(t)$  for all  $t \leq 0$ .

Define

$$\Psi(t) = \begin{cases} \Psi_+ & \text{if } t \geq 0, \\ \Psi_- & \text{if } t < 0. \end{cases}$$

Clearly  $\Psi(t)$  is a solution of (7) satisfying  $\Psi(-t) = \Psi(t)$  for all  $t \in \mathbb{R}$  and  $\Psi \rightarrow 0$  as  $t \rightarrow \infty$ . This completes the proof.  $\square$

The following results are analogous to Theorem 2.1 and Corollary 2.1.

**Theorem 4.1.** *Suppose that (C<sub>1</sub>), (C<sub>2</sub>), (C<sub>6</sub>), (C<sub>7</sub>) are valid. Then (1) and (2) are biasymptotically equivalent.*

**Corollary 4.1.** *Suppose that (C<sub>1</sub>), (C<sub>2</sub>), (C<sub>6</sub>), (C<sub>7</sub>) are valid, and that (2) has a  $k$ -parameter ( $k \leq n$ ) family  $v_1$  of almost periodic solutions. Then (1) admits a  $k$ -parameter family  $v_2$  of biasymptotically almost periodic solutions, and  $v_1$  is isomorphic to  $v_2$ .*

**Example 4.1.** Consider the system

$$y' = (A(t) + B(t))y, \quad (19)$$

where

$$A(t) = \begin{pmatrix} \sin \pi t & 0 \\ 0 & \sin \sqrt{5}t \end{pmatrix},$$

$B(t) = \cos t e^{-\alpha|t|} C$  with  $\alpha > 0$  a real number and  $C \in \mathbb{R}^{2 \times 2}$ .

Applying [Corollary 4.1](#) one can conclude that every solution of (19) is biasymptotically quasiperiodic.

## References

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