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The differential equations on time scales through impulsive differential equations

M.U. Akhmet[a,b,*], M. Turan[a]

[a] *Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey*
[b] *Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

Abstract

In this paper we investigate differential equations on certain time scales with transition conditions (DETC) on the basis of reduction to the impulsive differential equations (IDE). DETC are in some sense more general than dynamic equations on time scales [M. Bohner, A. Peterson, Dynamic equations on time scales, in: An Introduction With Applications, Birkhäuser Boston, Inc., Boston, MA, 2001, p. x+358; V. Laksmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamical Systems on Measure Chains, in: Math. and its Appl., vol. 370, Kluwer Academic, Dordrecht, 1996]. The basic properties of linear systems, the existence and stability of periodic solutions, and almost periodic solutions are considered. Appropriate examples are given to illustrate the theory.

MSC: 34A37; 34A30; 34C25; 34C27

Keywords: Differential equations on time scales; Impulsive differential equations; Periodic solutions; Almost periodic solutions; Stability

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1. Introduction

The theory of dynamic equations on time scales (DETS) has been developed over the last several decades [2,11,19]. After a literature survey about DETS, one can conclude that there are not so many results of the theory on the existence of periodic solutions and almost periodic

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[*] Corresponding author at: Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey. Tel.: +90 312 210 53 55; fax: +90 312 210 12 82.

E-mail addresses: marat@metu.edu.tr (M.U. Akhmet), meturan@metu.edu.tr (M. Turan).

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doi:10.1016/j.na.2005.12.042

solutions. Up to this moment, the investigations concerning linear DETS, integral manifolds and the stability of equations have not been fully developed. Certainly, these results should be obtained to be able to benefit from the applications of the theory. In our paper, we make an attempt to expand our knowledge of these aspects of the theory. We also propose a way to obtain these theoretical results. Moreover, we investigate differential equations on certain time scales with transition conditions (DETC), which are in some sense more general than DETS. At the same time, we should recognize that significant theoretical results concerning oscillations, boundary value problems, positive solutions, hybrid systems etc., have been achieved [1,2,8–15,17,19,20]. We assume that our proposals may stimulate new ideas by which the theory can also be developed, adding to previous significant achievements in this direction. The main idea of the paper is to apply the results of the theory of impulsive differential equations (IDE), the investigation of which started in the last century in the late 1960s [3–7,16,18,22]. We note that certain classes of DETC, particular with time scales, can be reduced to IDE, if we apply a special transformation [3] of the independent argument (the time variable). This transformation allows the reduced IDE to inherit all similar properties of the corresponding DETC. Then the investigation of the IDE can proceed using the known results. Finally, by taking the properties of the independent argument transformation into account, we can make an interpretation of the obtained results for DETC. The approach that we are using to connect the DETC with other types of differential equations is close to that of paper [20], where hybrid systems on time scales were considered.

This paper is organized as follows. In the next section, the time scale with its specific properties is considered. Moreover, the general form of DETC is described. The special transformation is given in Section 3. The reduction of DETC (DETS) to IDE is performed in Section 4. In Section 5, periodic solutions of linear equations and elements of Floquet's theory are considered, and the Massera theorem is proved. The last section is devoted to the problem of the existence and stability of almost periodic solutions.

2. Description of the DETC

Throughout the paper, we consider a specific time scale of the following type. Fix a sequence $\{t_i\} \in \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $|t_i| \to \infty$ as $|i| \to \infty$. Denote $\delta_i = t_{2i+1} - t_{2i}$, $\kappa_i = t_{2i} - t_{2i-1}$ and assume that:

(C0) $\sum_{-\infty} \kappa_i = \infty, \sum^{\infty} \kappa_i = \infty.$

The time scale $\mathbb{T}_0 = \bigcup_{i=-\infty}^{\infty} [t_{2i-1}, t_{2i}]$ is going to be considered throughout the paper.

Consider the following system of differential equations:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t, y), \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= J_i(y(t_{2i})) + y(t_{2i}), \end{aligned} \tag{2.1}$$

where the derivative is one sided at the boundary points of $\mathbb{T}_0$, $f(t, y) : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $J_i(y) : \mathbb{R}^n \to \mathbb{R}^n$, for all $i \in \mathbb{Z}$. We assume that functions f and J are continuous on their domains. More detailed characteristics of the functions will be given below when we consider specific problems. Let us introduce the following *transition operator*, $\Pi_i : \{t_{2i}\} \times \mathbb{R}^n \to \{t_{2i+1}\} \times \mathbb{R}^n$, $i \in \mathbb{Z}$, such that $\Pi_i(t_{2i}, y) = (t_{2i+1}, J_i(y) + y)$. Thus the evolution of the process is described by:

(1) the system of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad t \in \mathbb{T}_0; \tag{2.2}$$

(2) the transition operator Π_i, $i \in \mathbb{Z}$;
(3) the set $\mathbb{T}_0 \times \mathbb{R}^n$.

We shall call (2.1) the *differential equation on time scales with transition condition* (DETC). Let us show how to construct a solution of (2.1). Denote, by $\phi(t, \kappa, z)$, a solution of system (2.2) with an initial condition $\phi(\kappa, \kappa, z) = z, \kappa \in \mathbb{T}_0, z \in \mathbb{R}^n$, and, by $y(t)$, a solution of system (2.1) with an initial condition $y(t^0) = y_0$. Fix $t^0 \in \mathbb{T}_0$ such that $t_{2k-1} < t^0 < t_{2k}$ for some $k \in \mathbb{Z}$. If $t^0 \leq t < t_{2k}$, the solution is equal to $\phi(t, t^0, y_0)$, and $y(t_{2k}) = \phi(t_{2k}-, t^0, y_0)$, where the left limit is assumed to exist. Now, applying the transition operator, we obtain that $y(t_{2k+1}) = J_k(y(t_{2k})) + y(t_{2k})$. Thus the solution is not defined in the interval (t_{2k}, t_{2k+1}). Next, on the interval $[t_{2k+1}, t_{2(k+1)})$ the solution is equal to $\phi(t, t_{2k+1}, y(t_{2k+1}))$, and $y(t_{2(k+1)}) = \phi(t_{2(k+1)}-, t_{2k+1}, y(t_{2k+1}))$, and so on. If solution $y(t)$ is defined on a set $I \subset \mathbb{T}_0$, then the set $\{(t, y) : y = y(t), t \in I\}$ is called an *integral curve* of the solution.

Let us start with general information about differential equations on time scales. We provide only those facts of the theory that directly concern the needs of our paper. A more detailed description can be found in [2,11,19].

Any nonempty closed subset, $\mathbb{T}$, of $\mathbb{R}$ is called a time scale. On a time scale, the functions $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. The point $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$, and right-dense if $\sigma(t) = t$. Similarly, it is called left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Note that, on time scale $\mathbb{T}_0$, the points $t_{2i-1}, i \in \mathbb{Z}$, are left-scattered and right-dense, and the points $t_{2i}, i \in \mathbb{Z}$, are right-scattered and left-dense. Moreover, it is worth mentioning that $\sigma(t_{2i}) = t_{2i+1}, \rho(t_{2i+1}) = t_{2i}, i \in \mathbb{Z}$, and $\sigma(t) = \rho(t) = t$ for any other $t \in \mathbb{T}_0$.

The Δ-derivative of a continuous function f at a right-scattered point is defined as

$$f^{\Delta}(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

and at a right-dense point it is defined as

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if the limit exists.

Let $\mathbb{T}$ be an arbitrary time scale. A function $\varphi : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if:

(i) it is continuous at each right-dense or maximal $t \in \mathbb{T}$;
(ii) the left-sided limit $\varphi(t-) = \lim_{\xi \to t^-} \varphi(\xi)$ exists at each left-dense t.

Similarly, a function $\varphi : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if:

(i) it is continuous at each left-dense or minimal $t \in \mathbb{T}$;
(ii) the right-sided limit $\varphi(t+) = \lim_{\xi \to t^+} \varphi(\xi)$ exists at each right-dense t.

A differential equation

$$y^{\Delta}(t) = f(t, y), \quad t \in \mathbb{T} \tag{2.3}$$

is said to be a differential equation on time scale [19], where function $f(t, y) : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ in (2.3) is assumed to be rd-continuous on $\mathbb{T} \times \mathbb{R}^n$.

In our specific case we denote, by $\mathcal{T}_0$, the set of all functions which are rd-continuous on $\mathbb{T}_0$. Moreover, we define a set of functions $\mathcal{T}_0^1 \subset \mathcal{T}$ which are continuously differentiable on $\mathbb{T}_0$, assuming that the functions have a one-sided derivative at the boundary points of $\mathbb{T}_0$, that is, if

$\phi \in \mathcal{T}_0^1$, then $\phi' \in \mathcal{T}_0$. In general, by the derivative at the boundary point, we mean a one-sided derivative.

3. ψ-substitution

It is common to simplify a given equation by a proper transformation in every theory of differential equations. Likewise, in this section, we introduce a transformation which plays the role of a bridge in the passage from DETC, as in (2.1), to an IDE.

Without loss of generality, we assume that $t_{-1} < 0 < t_0$. The ψ-substitution, on the set $\mathbb{T}_0' = \mathbb{T}_0 \setminus \bigcup_{i=-\infty}^{\infty}\{t_{2i-1}\}$, is defined as

$$\psi(t) = \begin{cases} t - \sum\limits_{0<t_{2k}<t} \delta_k, & t \geq 0 \\ t + \sum\limits_{t\leq t_{2k}<0} \delta_k, & t < 0 \end{cases} \tag{3.4}$$

where $\delta_k = t_{2k+1} - t_{2k}$. Notice that the ψ-substitution is a one-to-one map, $\psi(0) = 0$, and condition (C0) implies that $\psi(\mathbb{T}_0') = \mathbb{R}$. The inverse transformation is

$$\psi^{-1}(s) = \begin{cases} s + \sum\limits_{0<s_k<s} \delta_k, & s \geq 0 \\ s - \sum\limits_{s\leq s_k<0} \delta_k, & s < 0. \end{cases} \tag{3.5}$$

Note that the inverse transformation is a piecewise continuous function with discontinuity of the first kind at the points $s = s_i$, $i \in \mathbb{Z}$, and $\psi^{-1}(s_i+) - \psi^{-1}(s_i) = \delta_i$.

Lemma 1. *$\psi'(t) = 1$ if $t \in \mathbb{T}_0'$.*

Proof. Assume that $t \geq 0$. Then,

$$\begin{aligned} \psi'(t) &= \lim_{h\to 0} \frac{\psi(t+h) - \psi(t)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\left(t + h - \sum_{0<t_{2k}<t+h} \delta_k\right) - \left(t - \sum_{0<t_{2k}<t} \delta_k\right)\right] \\ &= 1. \end{aligned}$$

The assertion for $t < 0$ can be proved in the same way.

Denote $s_i = \psi(t_{2i})$, $i \in \mathbb{Z}$. To make the reduction of DETC to IDE, we also need the following sets of functions. A function $\varphi(s) : \mathbb{R} \to \mathbb{R}^n$ is said to be in $\mathcal{PC}_0$ if:

(i) $\varphi(s)$ is left continuous on $\mathbb{R}$ and continuous on $\mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty}\{s_i\}$;
(ii) $\varphi(s)$ has discontinuities of the first kind at the points s_i.

Similarly, a function $\varphi(s)$ is said to be in $\mathcal{PC}_0^1$ if $\varphi \in \mathcal{PC}_0$ and φ' is in $\mathcal{PC}_0$, where

$$\varphi'(s_i) = \lim_{s\to s_i^-} \frac{\varphi(s) - \varphi(s_i)}{s - s_i}.$$

One can easily check that $\psi^{-1}(s) \in \mathcal{PC}_0^1$, and $\frac{\mathrm{d}}{\mathrm{d}s}(\psi^{-1}(s)) = 1$ if $s \neq s_i$, $i \in \mathbb{Z}$. □

In the next lemma, we show that the spaces of functions $\mathcal{T}_0$ and $\mathcal{PC}_0$ are closely related. This relation is set up by ψ-substitution. In the same manner, the relations between $\mathcal{T}_0^1$ and $\mathcal{PC}_0^1$ are going to be constructed. In what follows, assume that $s = \psi(t)$.

Lemma 2. *If* $\varphi(t) \in \mathcal{T}_0$, *then* $\varphi(\psi^{-1}(s)) \in \mathcal{PC}_0$, *and* $\varphi(\psi(t)) \in \mathcal{T}_0$ *if* $\varphi(s) \in \mathcal{PC}_0$.

Proof. Since $\psi(t)$ is a one-to-one transformation, we see that, if t is not one of the points t_k, then $\psi(t)$ is not one of the points s_i. Now, the continuity of ψ-substitution gives us the remaining part of the proof. □

Corollary 3. *If* $\varphi(t) \in \mathcal{T}_0^1$, *then* $\varphi(\psi^{-1}(s)) \in \mathcal{PC}_0^1$, *and* $\varphi(\psi(t)) \in \mathcal{T}_0^1$ *if* $\varphi(s) \in \mathcal{PC}_0^1$.

4. Reduction of the DETC (DETS) to impulsive differential equations

Using condition [19],

$$y^{\Delta}(t_{2i}) = \frac{y(t_{2i+1}) - y(t_{2i})}{t_{2i+1} - t_{2i}}, \quad i \in \mathbb{Z},$$

Eq. (2.3) can be written as

$$\begin{aligned} y'(t) &= f(t, y), \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= f(t_{2i}, y(t_{2i}))\delta_i + y(t_{2i}), \end{aligned} \tag{4.6}$$

where $\delta_i = t_{2i+1} - t_{2i}$.

We generalize the last equation if the specific term $f(t_{2i}, y(t_{2i}))\delta_i$ in (4.6) is replaced by an expression $J_i(y(t_{2i}))$, where J_i can be an arbitrary function.

Thus the following equation is considered:

$$\begin{aligned} y'(t) &= f(t, y), \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= J_i(y(t_{2i})) + y(t_{2i}). \end{aligned} \tag{4.7}$$

We name (4.7) as a *dynamic equation on time scale with transition condition*, and we abbreviate its name as DETC. Clearly, (4.6) is a specification of (4.7) with $J_i(y) = f(t, y)\delta_i$.

A function $\varphi \in \mathcal{T}_0^1$ is a solution of (4.7) if $\varphi'(t) = f(t, \varphi(t))$ for $t \in \mathbb{T}_0$, and $\varphi(t_{2i+1}) = J_i(\varphi(t_{2i})) + \varphi(t_{2i})$ for $t = t_{2i+1}$, $i \in \mathbb{Z}$.

Let us now apply the transformation of the independent argument $s = \psi(t)$ to (4.7). If $y(t)$ is a solution of (4.7), then $x(s) = y(\psi^{-1}(s))$ is a solution of the equation $x' = f(\psi^{-1}(s), x)$ for $s \neq s_i$, that is, $t \in \tilde{\mathbb{T}}_0$. If $t = t_{2i+1}$, then $s = \psi(t) = s_i^+$, and therefore the second equation in (4.7) leads to

$$x(s_i^+) = J_i(x(s_i)) + x(s_i),$$

which can be written as

$$\Delta x|_{s=s_i} = J_i(x(s_i)),$$

where $\Delta x|_{s=s_i} = x(s_i^+) - x(s_i)$. Thus, $x(s)$ is a solution of the following IDE:

$$\begin{aligned} x' &= f(\psi^{-1}(s), x), \quad s \neq s_i, \\ \Delta x|_{s=s_i} &= J_i(x(s_i)). \end{aligned} \tag{4.8}$$

The connection between DETC (4.7) and IDE (4.8) is established. The solution $x(s), x(s^0) = x_0, (s^0, x_0) \in (\mathbb{R} \times \mathbb{R}^n)$, of (4.8) satisfies the following integral equation:

$$x(s) = x_0 + \int_{s^0}^{s} f(\psi^{-1}(\xi), x(\xi))\,\mathrm{d}\xi + \sum_{s^0 \le s_i < s} J_i(x(s_i^+)), \tag{4.9}$$

if $s \ge s^0$, and

$$x(s) = x_0 + \int_{s^0}^{s} f(\psi^{-1}(\xi), x(\xi))\,\mathrm{d}\xi - \sum_{s \le s_i < s^0} J_i(x(s_i^+)), \tag{4.10}$$

if $s < s^0$.

Let a, b be members of $\mathbb{T}_0$ such that $a \le b$. We denote $\mathbb{T}_0(a,b) = [a, t_{2m}] \cup \sum_{k=m+1}^{p-1} [t_{2k-1}, t_{2k}] \cup [t_{2p-1}, b]$, where m and p are integers which satisfy $t_{2m-1} \le a \le t_{2m} < \cdots < t_{2p-1} \le t \le t_{2p}$, and for $f(\tau) \in \mathcal{T}_0$ we set

$$\int_{\mathbb{T}_0(a,b)} f(\tau)\,\mathrm{d}\tau := \int_a^{t_{2m}} f(\tau)\,\mathrm{d}\tau + \int_{t_{2m+1}}^{t_{2m+2}} f(\tau)\,\mathrm{d}\tau + \cdots + \int_{t_{2p-1}}^{b} f(\tau)\,\mathrm{d}\tau.$$

Now, the solution, $y(t), y(t^0) = y_0$, of (4.7), where $t^0 = \psi^{-1}(s^0)$, satisfies

$$y(t) = y_0 + \int_{\mathbb{T}_0(t^0,t)} f(\tau, y(\tau))\,\mathrm{d}\tau + \sum_{t^0 \le t_{2i} < t} J_i(y(t_{2i+1})), \tag{4.11}$$

if $t \ge t^0$, and

$$y(t) = y_0 - \int_{\mathbb{T}_0(t,t^0)} f(\tau, y(\tau))\,\mathrm{d}\tau - \sum_{t \le t_{2i} < t^0} J_i(y(t_{2i+1})) \tag{4.12}$$

if $t < t^0$.

5. Linear systems

5.1. Homogeneous linear systems

Let $f(t, y) = A(t)y$ and $J_i(y) = B_i y$ in (4.7), where $A(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $B_i \in \mathbb{R}^{n\times n}$. Consider the linear time scale differential equation

$$\begin{aligned} &y'(t) = A(t)y, \quad t \in \mathbb{T}_0, \\ &y(t_{2i+1}) = B_i y(t_{2i}) + y(t_{2i}). \end{aligned} \tag{5.13}$$

By means of ψ-substitution, system (5.13) turns out to be the IDE

$$\begin{aligned} &x' = \tilde{A}(s)x, \quad s \ne s_i, \\ &\Delta x|_{s=s_i} = B_i x, \end{aligned} \tag{5.14}$$

where $\tilde{A}(s) = A(\psi^{-1}(s))$. Since the solutions of system (5.14) form a linear space of dimension n [18,22], and ψ-substitution transforms only the time variable, the solutions of (5.13) also form a linear space of the same dimension, n.

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$ be the n-tuple whose j-th component is 1 and all others are 0, and assume that $x_j(s),\ x_j(0) = e_j$, is a solution of (5.14) for $j = 1, \ldots, n$. Then [22], for any other solution $x(s),\ x(0) = x_0$, of (5.14), we have

$$x(s) = \sum_{j=1}^{n} c_j x_j(s), \tag{5.15}$$

where the coefficients c_j are uniquely determined from $x_0 = \sum_{j=1}^{n} c_j e_j$.

Now, forming the matriciant $X(s) = \begin{bmatrix} x_1(s) & x_2(s) & \cdots & x_n(s) \end{bmatrix}$ of system (5.14), equality (5.15) can be written as

$$x(s) = X(s)x_0.$$

If $\mathcal{X}(s, r) = \mathcal{X}(s)\mathcal{X}^{-1}(r)$ is a transition matrix of $x' = \tilde{A}(s)x$, then

$$X(s) = \begin{cases} I, & s = 0 \\ \mathcal{X}(s,\ s_p)(I + B_p)\prod\limits_{k=p}^{1} \mathcal{X}(s_k,\ s_{k-1})(I + B_{k-1})\mathcal{X}(s_0, 0), & s > 0 \\ \mathcal{X}(s,\ s_l)(I + B_l)^{-1}\prod\limits_{k=l+1}^{-1} \mathcal{X}(s_{k-1},\ s_k)(I + B_k)^{-1}\mathcal{X}(s_{-1}, 0), & s < 0 \end{cases}$$

where, for $s > 0$, we have assumed that $0 < s_0 < \cdots < s_p < s < s_{p+1}$ and, for $s < 0$, we have assumed that $s_{l-1} < s < s_l < \cdots < s_{-1} < 0$.

On the other hand, ψ-substitution yields that a solution $y_j(t),\ y_j(0) = e_j$, is determined by

$$y_j(t) = x_j(\psi(t)).$$

Hence, any solution $y(t), y(0) = y_0$, of (5.13) is given by $y(t) = Y(t)y_0$, where the matriciant $Y(t)$ is defined and determined by

$$Y(t) = \begin{cases} I, & t = 0 \\ \mathcal{Y}(t, t_{2p+1})(I + B_p)\prod\limits_{k=p}^{1} \mathcal{Y}(t_{2k}, t_{2k-1})(I + B_{k-1})\mathcal{Y}(t_1, 0), & t > 0 \\ \mathcal{Y}(t, t_{2l})(I + B_l)^{-1}\prod\limits_{k=l+1}^{-1} \mathcal{Y}(t_{2k-1}, t_{2k})(I + B_k)^{-1}\mathcal{Y}(t_{-1}, 0), & t < 0 \end{cases}$$

in which $\mathcal{Y}(t, \tau) = \mathcal{Y}(t)\mathcal{Y}^{-1}(\tau)$ is a transition matrix of $y' = A(t)y$ and, for $t > 0$, we have assumed that $0 \le t_{2p+1} < t < t_{2(p+1)}$ and, for $t < 0$, we have assumed that $t_{2l-1} < t < t_{2l} \le 0$.

5.2. *Nonhomogeneous linear systems*

Consider the system

$$\begin{aligned} y'(t) &= A(t)y + g(t), \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + W_i + y(t_{2i}), \end{aligned} \tag{5.16}$$

where $y \in \mathbb{R}^n$, $A(t)$, B_i are as described for system (5.13), $g(t) \in \mathcal{T}_0$ and $\{W_i\}$, $i \in \mathbb{Z}$, is a sequence of n-vectors.

Applying the transformations $y(t) = Y(t)u(t)$ and $s = \psi(t)$, one can obtain

$$\begin{aligned} &z' = X^{-1}(s)\tilde{g}(s), \quad s \neq s_i, \\ &\Delta z|_{s=s_i} = X^{-1}(s_i^+)W_i \end{aligned} \tag{5.17}$$

where $z(s) = u(\psi^{-1}(s))$, $\tilde{g}(s) = g(\psi^{-1}(s))$. The solution of (5.17) satisfying $z(s^0) = z_0$ is

$$z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi)\tilde{g}(\xi)\,\mathrm{d}\xi + \sum_{s^0 \le s_i < s} X^{-1}(s_i^+)W_i, \tag{5.18}$$

if $s \geq s^0$, and

$$z(s) = z_0 + \int_{s^0}^{s} X^{-1}(\xi)\tilde{g}(\xi)\,\mathrm{d}\xi - \sum_{s \le s_i < s^0} X^{-1}(s_i^+)W_i, \tag{5.19}$$

if $s < s^0$. Consequently, the general solution of (5.16) is

$$y(t) = Y(t,t^0)y_0 + \int_{\mathbb{T}_0(t^0,t)} Y(t,\tau)g(\tau)\,\mathrm{d}\tau + \sum_{t^0 \le t_{2i} < t} Y(t,t_{2i+1})W_i, \tag{5.20}$$

if $t \geq t^0$, and

$$y(t) = Y(t,t^0)y_0 - \int_{\mathbb{T}_0(t,t^0)} Y(t,\tau)g(\tau)\,\mathrm{d}\tau - \sum_{t < t_{2i} \le t^0} Y(t,t_{2i+1})W_i, \tag{5.21}$$

if $t < t^0$.

5.3. Linear systems with constant coefficients

Let $A(t) \equiv A$ and $B_i \equiv B$ be constant matrices in (5.13) and consider the linear system with constant coefficients

$$\begin{aligned} &y' = Ay, \quad t \in \mathbb{T}_0, \\ &y(t_{2i+1}) = By(t_{2i}) + y(t_{2i}), \end{aligned} \tag{5.22}$$

where $A, B \in \mathbb{R}^{n \times n}$. The following assumptions for system (5.22) are needed:

(C1) the matrices A and B commute, $AB = BA$;
(C2) $\det(I + B) \neq 0$;
(C3) the limits

$$\lim_{t\to\infty} \frac{\psi(t) - \psi(t^0)}{t - t^0} = \ell, \quad \lim_{t\to\infty} \frac{i(t^0,t)}{t - t^0} = p$$

exist, where $i(t^0,t)$ is the number of gaps, (t_{2k}, t_{2k+1}), in $\mathbb{T}_0$ between t^0 and t.

Denote $\Lambda_0 = \ell A + p \ln(I + B)$.

Theorem 4. *Let conditions* (C0)–(C3) *hold. Then the zero solution of* (5.22) *is*

(a) *asymptotically stable if the real parts of all eigenvalues of the matrix Λ_0 are negative;*

(b) *unstable if the real part of at least one eigenvalue of the matrix Λ_0 is positive.*

Proof. It is easily seen that $\mathcal{Y}(t,\tau) = \mathrm{e}^{A(t-\tau)}$ and hence, if $t_{2m-1} \le t^0 \le t_{2m} < \cdots < t_{2p-1} \le t \le t_{2n}$, we get

$$Y(t,t^0) = \mathrm{e}^{A(t-t_{2n-1})}(I+B)\prod_{k=n-1}^{m+1}\left[\mathrm{e}^{A(t_{2k}-t_{2k-1})}(I+B)\right]\mathrm{e}^{A(t_{2m}-t^0)}.$$

Condition (C1) implies that $Y(t,t^0) = \mathrm{e}^{A[\psi(t)-\psi(t^0)]}(I+B)^{i(t^0,t)}$. Due to condition (C3), we can write

$$\psi(t)-\psi(t^0) = [\ell+\epsilon_1(t)](t-t^0), \quad \text{and,} \quad i(t^0,t) = [p+\epsilon_2(t)](t-t^0)$$

where $\epsilon_j(t) \to 0$ as $t \to \infty$, $j = 1,2$. In general, the functions $\epsilon_j(t)$, $j = 1,2$, are piecewise continuous functions.

Now, the solution $y(t), y(t^0) = y_0$, of (5.22) can be written as $y(t) = \mathrm{e}^{\Lambda(t)(t-t^0)}y_0$, where $\Lambda(t) = \Lambda_0 + \epsilon_1(t)A + \epsilon_2(t)\ln(I+B)$ for $t \ge t^0$.

Assume that $\max_j \operatorname{Re}\lambda_j(\Lambda_0) = \gamma < 0$. The properties of functions ϵ_j, $j = 1,2$, imply that, for a fixed positive ϵ, there exists a sufficiently large $T > 0$ such that, if $t \ge T$, then $|\epsilon_j(t)| < \epsilon$, $j = 1,2$.

Therefore,

$$\|y(t)\| \le K(\bar{\epsilon})\mathrm{e}^{\varkappa(\epsilon)(t-t^0)}\mathrm{e}^{(\gamma+\bar{\epsilon})(t-t^0)},$$

where $\varkappa(\epsilon) = \|\epsilon_1(t)A + \epsilon_2(t)\ln(I+B)\|$. Since $\gamma < 0$ and $\epsilon, \bar{\epsilon}$ can be chosen so small that $\gamma + \bar{\epsilon} + \varkappa(\epsilon) < 0$, the first part of the theorem is proved.

Let λ_0 be the eigenvalue of Λ_0, whose real part is positive, and let y_0 be a corresponding eigenvector in a small neighborhood of the origin. We can obtain that

$$\|y(t)\| \ge \mathrm{e}^{-\varkappa(\epsilon)(t-t^0)}\mathrm{e}^{\operatorname{Re}\lambda_0(t-t^0)}\|y_0\|.$$

Since $\operatorname{Re}\lambda_0 > 0$, we can choose $\epsilon > 0$ so small that $-\varkappa(\epsilon) + \operatorname{Re}\lambda_0 > 0$, and the last inequality completes the proof. □

Example 5.1. Let $t_i = i + (-1)^i\kappa$, $0 < \kappa \le \frac{1}{3}$, and consider the system

$$\begin{aligned} y_1' &= \alpha y_1 - \beta y_2, \\ y_2' &= \beta y_1 + \alpha y_2, \quad t \in \mathbb{T}_0, \\ y_1(t_{2i+1}) &= (1+k)y_1(t_{2i}), \\ y_2(t_{2i+1}) &= (1+k)y_2(t_{2i}), \end{aligned} \tag{5.23}$$

where β is a positive real number and $k > -1$ is a constant. One can easily see that the matrices $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ commute with each other and $\ell = \frac{1}{2} + \kappa$, $p = \frac{1}{2}$. Therefore, we have

$$\Lambda_0 = \begin{bmatrix} \left(\frac{1}{2}+\kappa\right)\alpha + \frac{1}{2}\ln(1+k) & -\left(\frac{1}{2}+\kappa\right)\beta \\ \left(\frac{1}{2}+\kappa\right)\beta & \left(\frac{1}{2}+\kappa\right)\alpha + \frac{1}{2}\ln(1+k) \end{bmatrix}$$

which has eigenvalues $\lambda_{1,2} = (\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) \pm (\frac{1}{2}+\kappa)\beta i$. Hence, the zero solution of (5.23) is asymptotically stable if $(\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) < 0$; unstable if $(\frac{1}{2}+\kappa)\alpha + \frac{1}{2}\ln(1+k) > 0$.

6. Periodic solutions

6.1. Description of periodical DETC

Definition 5. The time scale $\mathbb{T}_0$ is said to have an ω-property if there exists a number $\omega \in \mathbb{R}^+$ such that $t + \omega \in \mathbb{T}_0$ whenever $t \in \mathbb{T}_0$.

From this definition, by simply using mathematical induction, we get the following lemma.

Lemma 6. *If $\mathbb{T}_0$ has an ω-property, then $t + n\omega \in \mathbb{T}_0$ for all $t \in \mathbb{T}_0, n \in \mathbb{Z}$.*

Definition 7. A sequence $\{a_i\} \subset \mathbb{R}$ is said to satisfy an (ω, p)-property if there exist numbers $\omega \in \mathbb{R}^+$ and $p \in \mathbb{N}$ such that $a_{i+p} = a_i + \omega$ for all $i \in \mathbb{Z}$.

Lemma 8. *If t is a right(left)-dense point of $\mathbb{T}_0$ which has an ω-property, then $t + n\omega$ is also a right(left)-dense point, respectively, $\forall n \in \mathbb{Z}$.*

Proof. We will prove the statement just for $n = 1$, since the remaining part is an obvious application of mathematical induction. Let t be a right-dense point. Then,

$$\begin{aligned}\sigma(t+\omega) &= \inf\{s > t + \omega : s \in \mathbb{T}_0\} = \inf\{s > t : s \in \mathbb{T}_0\} + \omega \\ &= \sigma(t) + \omega = t + \omega,\end{aligned}$$

that is, $t + \omega$ is a right-dense point. Similarly, one can prove the lemma for left-dense points. □

Corollary 9. *If $\mathbb{T}_0$ has an ω-property, then there exists $p \in \mathbb{N}$ such that the sequences $\{t_{2i}\}$ and $\{t_{2i+1}\}$ satisfy an (ω, p)-property.*

Corollary 10. *If $\mathbb{T}_0$ has an ω-property, the sequence $\{\delta_k\}$ is p-periodic, that is, $\delta_{k+p} = \delta_k$ for all $k \in \mathbb{Z}$.*

Let p_0 be the minimal of these numbers p which satisfy Corollary 10.

Lemma 11. *If $\mathbb{T}_0$ has an ω-property, then the sequence $\{s_i\}$, $s_i = \psi(t_{2i})$, is $(\tilde{\omega}, p_0)$-periodic with*

$$\tilde{\omega} = \omega - \sum_{0<t_{2k}<\omega} \delta_k = \psi(\omega).$$

Proof. In order to prove this lemma, we only need to verify that $s_{i+p_0} = s_i + \tilde{\omega}$ for all i. Assume that $i \geq 0$, $i = np_0 + j$ for some $n \in \mathbb{Z}$, $0 \leq j < p_0$ and $0 < t_0 < \cdots < t_{2(p_0-1)} < \omega$. Then

$$\begin{aligned} s_{i+p_0} &= \psi(t_{2(i+p_0)}) = t_{2(i+p_0)} - \sum_{0<t_{2k}<t_{2(i+p_0)}} \delta_k \\ &= t_{2i} + \omega - \sum_{0<t_{2k}<t_{2i}} \delta_k - \sum_{t_{2i}\leq t_{2k}<t_{2(i+p_0)}} \delta_k = \psi(t_{2i}) + \omega - \sum_{k=i}^{i+p_0-1} \delta_k \\ &= s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_{k+np_0} = s_i + \omega - \sum_{k=j}^{j+p_0-1} \delta_k = s_i + \omega - \sum_{k=0}^{p_0-1} \delta_k \\ &= s_i + \omega - \sum_{0<t_{2k}<\omega} \delta_k = s_i + \tilde{\omega}, \end{aligned}$$

where we have used the fact that

$$\sum_{k=j}^{j+p_0-1} \delta_k = \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=p_0}^{j+p_0-1} \delta_k = \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_{k+p_0}$$
$$= \sum_{k=j}^{p_0-1} \delta_k + \sum_{k=0}^{j-1} \delta_k = \sum_{k=0}^{p_0-1} \delta_k.$$

All other cases can be verified similarly. □

Corollary 12. *If* $\mathbb{T}_0$ *has an* ω*-property, then* $\psi(t+\omega) = \psi(t) + \psi(\omega)$.

Denote the set of all T-periodic functions, defined on the set $\mathcal{A} \subset \mathbb{R}$, by $\mathcal{P}_T(\mathcal{A})$.

Lemma 13. *If* $\phi(t) \in \mathcal{P}_\omega(\mathbb{T}_0)$, *and* $\mathbb{T}_0$ *has an* ω*-property, then* $\phi(\psi^{-1}(s)) \in \mathcal{P}_{\tilde{\omega}}(\mathbb{R})$ *with* $\tilde{\omega} = \psi(\omega)$.

Proof. By Corollary 12, $s + \tilde{\omega} = \psi(t+\omega)$. Then the equality

$$\phi(\psi^{-1}(s+\tilde{\omega})) = \phi(t+\omega) = \phi(t) = \phi(\psi^{-1}(s))$$

completes the proof. □

Similarly, to the last lemma, the following assertion can be verified.

Lemma 14. *If* $\phi(s) \in \mathcal{P}_{\tilde{\omega}}(\mathbb{R})$, *then* $\phi(\psi(t)) \in \mathcal{P}_\omega(\mathbb{T}_0)$.

6.2. *Floquet theory*

Consider

$$\begin{aligned} y'(t) &= A(t)y + f(t), \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + J_i + y(t_{2i}), \end{aligned} \tag{6.24}$$

where $A, f \in \mathcal{P}_\omega(\mathbb{T}_0)$, sequences B_i and J_i are p-periodic, $\mathbb{T}_0$ has an ω-property, and let $Y(t),\ Y(0) = I$, be the fundamental matrix solution of the corresponding homogeneous system

$$\begin{aligned} y'(t) &= A(t)y, \quad t \in \mathbb{T}_0, \\ y(t_{2i+1}) &= B_i y(t_{2i}) + y(t_{2i}). \end{aligned} \tag{6.25}$$

A solution $y(t),\ y(t^0) = y_0$, of (6.24) is given by

$$y(t) = Y(t)y_0 + \int_{\mathbb{T}_0(0,t)} Y(t,\tau) f(\tau)\,\mathrm{d}\tau + \sum_{0<t_{2i}<t} Y(t, t_{2i+1}) J_i.$$

Now, for this solution to be ω-periodic, we need $y(\omega) = y(0) = y_0$, that is,

$$[I - Y(\omega)]y_0 = b \tag{6.26}$$

where

$$b = \int_{\mathbb{T}_0(0,\omega)} Y(\omega,\tau) f(\tau)\,\mathrm{d}\tau + \sum_{0<t_{2i}<\omega} Y(\omega, t_{2i+1}) J_i. \tag{6.27}$$

Definition 15. The eigenvalues, ρ_j, of the matrix $Y(\omega)$ are called Floquet multipliers (or simply multipliers) of system (6.24).

The following Theorems 16–18 can be proved as similar assertions for ordinary differential equations.

Theorem 16. *If ρ is a multiplier, then there exists a nontrivial solution, $y(t)$, of (6.25) such that $y(t+\omega) = \rho y(t)$. Conversely, if there exists a nontrivial solution, $y(t)$, of (6.25) such that $y(t+\omega) = \rho y(t)$, then ρ is a multiplier.*

Theorem 17. *System (6.25) has a $k\omega$-periodic solution if and only if there exists a multiplier, ρ, such that $\rho^k = 1$.*

Now, if we have $\rho \neq 1$ for all multipliers, then the system in (6.26) has a unique solution which implies the following theorem.

Theorem 18. *If unity is not one of the multipliers, then (6.24) has a unique ω-periodic solution, $y(t)$, such that $y(0) = y_0 = [I - Y(\omega)]^{-1}b$.*

Now, we can write the matriciant, $Y(t)$, in the Floquet form

$$Y(t) = \Phi(t)\mathrm{e}^{P\psi(t)}$$

where $\Phi(t) = Y(t)\mathrm{e}^{-P\psi(t)}$, $P = \frac{1}{\tilde{\omega}} \ln Y(\omega)$, $\tilde{\omega} = \psi(\omega)$. Then

$$\begin{aligned}\Phi(t+\omega) &= Y(t+\omega)\mathrm{e}^{-P\psi(t+\omega)} = Y(t)Y(\omega)\mathrm{e}^{-P\psi(\omega)}\mathrm{e}^{-P\psi(t)} \\ &= Y(t)\mathrm{e}^{-P\psi(t)} = \Phi(t)\end{aligned}$$

and hence $\Phi(t)$ is ω-periodic. From the definition of $\Phi(t)$, we see that $\Phi(t)$ is continuously differentiable, bounded for all $t \in \mathbb{T}_0$ because of its periodicity, and is nonsingular. One can easily verify that the transformation $y = \Phi(t)u$ transforms system (6.25) into the system with constant coefficients

$$\begin{aligned}&u' = Pu, \quad t \in \mathbb{T}_0 \\ &u(t_{2i+1}) = u(t_{2i}),\end{aligned} \tag{6.28}$$

where we have used the fact that

$$\mathrm{e}^{-P\psi(t_{2i+1})} = \mathrm{e}^{-P\psi(t_{2i})}.$$

Definition 19. The eigenvalues, λ_j, of the matrix $P = \frac{1}{\tilde{\omega}} \ln Y(\omega)$ are called the Floquet exponents (or simply exponents).

Similarly to ODE and applying the Floquet theory for IDE [22], one can prove that the following assertions are valid.

Theorem 20. *Let $\{\lambda_j\}$ be the exponents. Then the solutions of (6.25) are*

(a) *asymptotically stable if and only if* $\operatorname{Re}(\lambda_j) < 0$ *for all* j;
(b) *stable if* $\operatorname{Re}(\lambda_j) \leq 0$ *for all* j *and* λ_j *is simple when* $\operatorname{Re}\lambda_j = 0$;
(c) *unstable if there exists an exponent* λ_j *such that* $\operatorname{Re}(\lambda_j) > 0$.

Theorem 21. *Let $\{\rho_j\}$ be the multipliers. Then the solutions of (6.25) are*

(a) *asymptotically stable if and only if all multipliers lie inside the unit circle;*
(b) *stable if* $|\rho_j| \leq 1$ *for all* j *and* ρ_j *is simple when* $|\rho_j| = 1$*;*
(c) *unstable if there exists a multiplier* ρ_j *which lies outside the unit circle.*

Example 6.1. Let $t_i = \mathrm{i}\pi + (-1)^i \frac{\pi}{4}$ and consider the system

$$\begin{aligned} y_1' &= -y_2 + f_1(t), \\ y_2' &= y_1 + f_2(t), \quad t \in \mathbb{T}_0, \\ y_1(t_{2i+1}) &= (1+k)y_1(t_{2i}), \\ y_2(t_{2i+1}) &= (1+k)y_2(t_{2i}), \end{aligned} \tag{6.29}$$

where $f_1(t) = \mathrm{e}^{t-t_{2i-1}}$ and $f_2(t) = \sin(t - t_{2i-1})$ for $t_{2i-1} < t \leq t_{2i}$ and $k \in \mathbb{R}$ is a constant. It is easy to see that this system is 2π-periodic and the matriciant of the corresponding homogeneous system is

$$\mathcal{Y}(t,\tau) = \begin{bmatrix} \cos(t-\tau) & -\sin(t-\tau) \\ \sin(t-\tau) & \cos(t-\tau) \end{bmatrix}$$

and hence the matrix of monodromy is

$$Y(2\pi) = \mathcal{Y}\left(2\pi, \frac{3\pi}{4}\right)(I+B)\mathcal{Y}\left(\frac{\pi}{4}, 0\right) = \begin{bmatrix} 1+k & 0 \\ 0 & 1+k \end{bmatrix}.$$

Therefore, the multipliers are $\rho_{1,2} = 1+k$. Now, if $k \neq 0$ then, by Theorem 18, the system in (6.29) has a unique 2π-periodic solution and, by Theorem 21, this periodic solution is asymptotically stable for $-2 < k < 0$, unstable for $k < -2$ or $k > 0$, and stable for $k = -2$.

6.3. *The Massera theorem*

Let us consider the following analogue of the famous theorem from [21].

Theorem 22. *If system* (6.24) *has a bounded solution* $y^*(t)$ *on the set* $\{t \in \mathbb{T}_0 : t \geq 0\}$*, then there exists a periodic solution of system* (6.24).

Proof. Assume on the contrary that there exists no periodic solution. Let $y^*(t)$, $y(0) = y_0$, be a bounded solution of (6.24), then

$$y^*(t) = Y(t)y_0 + \int_{\mathbb{T}_0(0,t)} Y(t)Y^{-1}(\tau)f(\tau)\,\mathrm{d}\tau + \sum_{0<t_{2i}<t} Y(t)Y^{-1}(t_{2i+1})J_i$$

and $y^*(\omega) = Y(\omega)y_0 + b$, where b is as in (6.27). Now, $x^*(s) = y^*(\psi^{-1}(s))$ is a solution of

$$\begin{aligned} x' &= A(\psi^{-1}(s))x + f(\psi^{-1}(s)), \quad s \neq s_i \\ \Delta x|_{s=s_i} &= B_i x + J_i. \end{aligned} \tag{6.30}$$

Since $x^*(s+\tilde{\omega}) = y^*(\psi^{-1}(s+\tilde{\omega})), \tilde{\omega} = \psi(\omega)$, is also a solution of (6.30), it implies that $y^*(t+\omega)$ is also a solution of (6.24). Thus, we have

$$\begin{aligned} y^*(t+\omega) = {} & Y(t+\omega)y_0 + \int_{\mathbb{T}_0(0,t+\omega)} Y(t+\omega)Y^{-1}(\tau)f(\tau)\,\mathrm{d}\tau \\ & + \sum_{0<t_{2i}<t+\omega} Y(t+\omega)Y^{-1}(t_{2i+1})J_i \end{aligned}$$

$$\begin{aligned}&= Y(t)y^*(\omega) + \int_{\mathbb{T}_0(0,t)} Y(t)Y^{-1}(\tau)\, f(\tau)\,\mathrm{d}\tau \\ &\quad + \sum_{0<t_{2i}<t} Y(t)Y^{-1}(t_{2i+1})J_i\end{aligned}$$

and

$$y^*(2\omega) = Y(\omega)y^*(\omega) + b = Y^2(\omega)y_0 + Y(\omega)b + b.$$

Continuing in this way, by mathematical induction we see that

$$y^*(n\omega) = Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b.$$

If there is no ω-periodic solution, then the system $[I - Y(\omega)]y_0 = b$ has no solution, which means that there is a solution, c, of the system $[I - Y(\omega)]^{\mathrm{T}}y = 0$ such that $\langle b, c\rangle \neq 0$. Thus,

$$\begin{aligned}\langle y^*(n\omega), c\rangle &= \left\langle Y^n(\omega)y_0 + \sum_{k=0}^{n-1} Y^k(\omega)b, c\right\rangle \\ &= \langle y_0, [Y^n(\omega)]^{\mathrm{T}}c\rangle + \sum_{k=0}^{n-1}\langle b, [Y^k(\omega)]^{\mathrm{T}}c\rangle = \langle y_0, c\rangle + \sum_{k=0}^{n-1}\langle b, c\rangle \\ &= \langle y_0, c\rangle + n\langle b, c\rangle\end{aligned}$$

which becomes unbounded as $n \to \infty$. However, since $y^*(t)$ is bounded, we have

$$|\langle y^*(n\omega), c\rangle| \leq |y^*(n\omega)||c| \leq M|c|$$

and this contradiction completes the proof. □

Corollary 23. *If system (6.24) does not have an ω-periodic solution, then all solutions of system (6.24) are unbounded on both $\{t \in \mathbb{T}_0 : t \geq 0\}$ and $\{t \in \mathbb{T}_0 : t < 0\}$.*

7. Almost periodic solutions of quasi-linear systems

Let us consider the following quasilinear DETC:

$$\begin{aligned}&y' = A(t)y + f(t, y), \quad t \in \mathbb{T}_0, \\ &y(t_{2i+1}) = B_i y(t_{2i}) + J_i(y(t_{2i})) + y(t_{2i}).\end{aligned} \tag{7.31}$$

The definition of almost periodic functions on time scales will be introduced and the conditions under which the system admits this type of almost periodic solutions will be obtained.

Fix a number $\theta > 0$. Let Θ be a set of sequences $\{\theta_i\} \subset R$, $i \in \mathbb{Z}$, such that $\theta_{i+1} - \theta_i \geq \theta$, $i \in \mathbb{Z}$. From the last inequality, it implies that $|\theta_i| \to \infty$, as $|i| \to \infty$. Let us introduce the following definitions.

An integer p is called an ϵ-almost period of a sequence of vectors $\{a_i\}$, $i \in \mathbb{Z}$, if $\|a_{i+p} - a_i\| < \epsilon$ for any $i \in \mathbb{Z}$.

Definition 24. A sequence a_i, $i \in \mathbb{Z}$, is almost periodic if, for any $\epsilon > 0$, there exists a relatively dense set of its ϵ-almost periods.

Denote $\theta_i^j = \theta_{i+j} - \theta_i$ and define sequences $\theta^j = \{\theta_i^j\}_i$, $j \in \mathbb{Z}$.

A set $S \subset R$ is said to be relatively dense if there exists a number $\ell > 0$ such that $[a, a+\ell] \cap S \neq \emptyset$ for all $a \in R$.

Definition 25. [16,22] We call $\{\theta_i\}$ a Wexler sequence if sequences θ^j, $j \in \mathbb{Z}$, are equipotentially almost periodic, that is, for an arbitrary $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods that are common for all θ^j, $j \in \mathbb{Z}$.

It is obvious that every (ω, p)-periodic sequence is also a Wexler sequence. Let us again use the sequence $\{s_i\}$, where $s_i = \psi(t_{2i})$, $j \in \mathbb{Z}$. One can easily check that $\{s_i\} \in \Theta$. In this section, we assume that

(C4) both $\{s_i\}$ and $\{t_i\}$ are Wexler sequences.

Fix $H > 0$, and denote $G_H = \{x \in \mathbb{R}^n : \|x\| \leq H\}$. Let $\mathcal{T}_0(G_H)$ be a set of functions $f(t,x) : \mathbb{T}_0 \times G_H \to \mathbb{R}^n$ such that $f(t,x) \in \mathcal{T}_0$ for every fixed $x \in G_H$. For $f \in \mathcal{T}_0$ (respectively, $f \in \mathcal{T}_0(G_H)$) and $\tau \in R$, the translate of f by τ is the function $Q_\tau f = f(t+\tau)$, $t \in \mathbb{T}_0$ (respectively, $Q_\tau f(t,x) = f(t+\tau, z)$, $(x,t) \in \mathbb{T}_0 \times G_H$). A number $\tau \in R$ is called an ϵ-translation number of a function $f \in \mathcal{T}_0$ $(f \in \mathcal{T}_0(G_H))$ if $t + \tau \in \mathbb{T}_0$ for all $|t - t_i| > \epsilon$ and $\|Q_\tau f - f\| < \epsilon$ for every $t \in \mathbb{T}_0((t,z) \in \mathbb{T}_0 \times G_H)$, such that $|t - t_i| > \epsilon$. It is obvious that the function $Q_\tau f$ is not defined if $t + \tau \notin \mathbb{T}_0$.

Definition 26. A function $f \in \mathcal{T}_0(f \in \mathcal{T}_0(G_H))$ is called almost periodic on $\mathbb{T}_0$ (almost periodic in t uniformly with respect to $x \in G_H$) if, for every $\epsilon \in R, \epsilon > 0$, there exists a respectively dense set of ϵ-translations of f.

Denote by $\mathcal{AP}(\mathbb{T}_0)$ $(\mathcal{AP}(\mathbb{T}_0 \times G_H))$ the set of all such functions.

Let $\mathcal{PC}_0(G_H)$ be a set of functions $f(s,x) : \mathbb{R} \times G_H \to \mathbb{R}^n$ such that $f(s,x) \in \mathcal{PC}_0$ for every fixed $x \in G_H$. For $f \in \mathcal{PC}_0$ (respectively, $f \in \mathcal{PC}_0(G_H)$) and $\tau \in R$, the translate of f by τ is the function $Q_\tau f = f(s+\tau), s \in \mathbb{R}$ (respectively, $Q_\tau f(s,x) = f(s+\tau, z)$, $(s,x) \in \mathbb{R} \times G_H$). A number $\tau \in R$ is called an ϵ-translation number of a function $f \in \mathcal{PC}_0$ $(f \in \mathcal{PC}_0(G_H))$ if $\|Q_\tau f - f\| < \epsilon$ for every $s \in \mathbb{R}((s,x) \in \mathbb{R} \times G_H)$, such that $|s - s_i| > \epsilon$.

Definition 27. [16,22] A function $f \in \mathcal{PC}_0(f \in \mathcal{PC}_0(G_H))$ is called almost periodic (almost periodic in s uniformly with respect to $x \in G_H$) if, for every $\epsilon \in R, \epsilon > 0$, there exists a respectively dense set of ϵ-translations of f.

Denote by $\mathcal{BWAP}$ $(\mathcal{BWAP}(G_H))$ the set of all such functions.

Lemma 28. [22] *If $\{s_i\}$ is a Wexler sequence and if $\phi \in \mathcal{BWAP}$, then, for arbitrary $\epsilon > 0, \epsilon > \nu > 0$, there exists $l > 0$ such that, for an arbitrary interval J of length l, there exists $\tau \in J$ and $q_\tau \in \mathbb{Z}$, such that*

(a) $|s_i^{q_\tau} - \tau| < \epsilon, i \in \mathbb{Z}$;
(b) $\|\phi(s+\tau) - \phi(s)\| < \epsilon$, *if* $|s - s_i| > \epsilon$, $i \in \mathbb{Z}$, $s \in \mathbb{R}$.

Let us prove the following important lemma.

Lemma 29. *If $\phi \in \mathcal{BWAP}$, then $\phi(\psi(t)) \in \mathcal{AP}(\mathbb{T}_0)$.*

Proof. Assume that $\phi(s) \in \mathcal{B}WAP$. Fix $\epsilon > 0$. It is known [16,22] that ϕ is uniformly continuous on the union of intervals of continuity $(s_i,\ s_{i+1}), i \in \mathbb{Z}$. Denote $\delta > 0$ a positive number such that $\|\phi(s) - \phi(s')\| < \epsilon$ if $|s - s'| < \delta$ and $s,\ s'$ belong to one and the same interval of continuity of ϕ. Without loss of generality, we assume that $\tau > 0,\ s > 0$, and denote $\tau' = \tau + \sum_{0<s_i<\tau} \delta_i$, $s = \psi(t)$. We have that

$$\begin{aligned}\psi(t+\tau') - s - \tau &= t + \tau + \sum_{0<s_i<\tau} \delta_i - \sum_{0<t_{2i}<t+\tau'} \delta_i - t + \sum_{0<t_{2i}<t} \delta_i - \tau \\ &= \sum_{0<s_i<\tau} \delta_i - \sum_{t\le t_{2i}<t+\tau} \delta_i .\end{aligned}$$

Denote $\nu = \max\{\frac{\epsilon}{4}, \delta\}$. Using the method of the proof of Lemma 28 [16,22] (the method of common almost periods), we can prove that there exist relatively dense sets of numbers τ and q_τ such that

(1) $\|\phi(s+\tau) - \phi(s)\| < \epsilon$ if $|s - s_i| > \epsilon, i \in \mathbb{Z},\ s \in \mathbb{R}$;
(2) $|s_i^{q_\tau} - \tau| < \nu, i \in \mathbb{Z}$;
(3) $|\sum_{0<s_i<\tau} \delta_i - \sum_{t\le t_{2i}<t+\tau} \delta_i| < \nu$.

Then it implies that $|s + \tau - s_i| > \nu$ and $s + \tau$, $\psi(t + \tau')$ are in one and the same interval of continuity of ϕ. Now we can conclude that

$$\begin{aligned}\|\phi(\psi(t+\tau')) - \phi(\psi(t))\| &= \|\phi(\psi(t+\tau')) - \phi(s+\tau)\| + \|\phi(s+\tau) - \phi(\psi(t))\| \\ &= \|\phi(\psi(t+\tau')) - \phi(s+\tau)\| + \|\phi(s+\tau) - \phi(s)\| < 2\epsilon.\end{aligned}$$

The density of numbers τ' follows the density of the set $\{\tau\}$. The lemma is proved. □

Similarly, we can prove that the following lemma is valid.

Lemma 30. *If* $f(t) \in \mathcal{A}P(\mathbb{T}_0)$, *then* $f(\psi^{-1}(s)) \in \mathcal{B}WAP$.

The last lemma implies that every function from $\mathcal{A}P(\mathbb{T}_0)$ is uniformly continuous on $\mathbb{T}_0$. Let us make the following assumptions for system (7.31):

(C5) elements of matrix $A(t)$ are from $\mathcal{A}P(\mathbb{T}_0)$, elements of function $f(t,x)$ are from $\mathcal{A}P(\mathbb{T}_0 \times G_H)$;
(C6) sequence $B_i, i \in \mathbb{Z}$, is almost periodic, and sequence $J_i, i \in \mathbb{Z}$, is almost periodic uniformly with respect to $x \in G_H$;
(C7) functions f and J_i satisfy the Lipschitz condition with constant L;
(C8) There exist positive constants K and α such that

$$\|Y(t,\xi)\| \le K\mathrm{e}^{-\alpha(t-\xi)}, \quad t \ge \xi, t, \xi \in \mathbb{T}_0, \tag{7.32}$$

where $Y(t,\ s)$ is the transition matrix of homogeneous system (5.13).

By using ψ-substitution, we can write that

$$\|X(s,\eta)\| \le K\mathrm{e}^{-\alpha(s-\eta+\sum\limits_{\eta\le s_i<s}\delta_i)}, \quad s \ge \eta. \tag{7.33}$$

Hence,

$$\|X(s,\eta)\| \le K_1\mathrm{e}^{-2\alpha(s-\eta)}, \quad s \ge \eta \tag{7.34}$$

where $K_1 = K\mathrm{e}^{\alpha\theta}$.

Now, consider the following impulsive system

$$\begin{aligned} x'(s) &= A(\psi^{-1}(s))x + f(\psi^{-1}(s), x(s)), \quad s \neq s_i \\ \Delta x|_{s=s_i} &= B_i x(s_i) + J_i(x(s_i)), \end{aligned} \tag{7.35}$$

which corresponds to (7.31) by ψ-substitution. It implies from Lemma 30 that $A(\psi^{-1}(s)) \in \mathcal{WBAP}$ and $f(\psi^{-1}(s), x) \in \mathcal{WBAP}(G_H)$. By Lemma 26 [22], there exists a number $N \in \mathbb{N}$ such that every unit interval of $\mathbb{R}$ contains not more than N elements of sequence s_i. Denote

$$\begin{aligned} a &= \frac{1}{2\alpha} + \frac{e^{2(\frac{1}{N}-1)\alpha}}{1 - e^{2\alpha\frac{1}{N}}}, \\ M &= \sup_{\mathbb{T}_0 \times G_H} \|f(t, y)\| + \sup_{\mathbb{Z} \times G_H} \|J_i(y)\|. \end{aligned}$$

Now, using Theorem 82 from [22] (see, also, [7]), we can conclude that the following lemma is valid.

Lemma 31. *Assume that conditions* (C4)–(C8) *and* (7.34) *are fulfilled and, moreover:*

(1) $K_1 Ma < H$;
(2) $K_1 La < 1$;
(3) $2\alpha + K_1 L + N^{-1} \ln(1 + K_1 L) < 0$.

Then system (7.35) *has a unique asymptotically stable solution* $x_0(s) \in \mathcal{WBAP}$.

Applying the properties of ψ-substitution and Lemmas 29 and 31, we can formulate the following theorem.

Theorem 32. *Assume that conditions* (C4)–(C8) *and* (7.32) *are fulfilled and, moreover:*

(1) $Ke^{\alpha\theta} Ma < H$;
(2) $Ke^{\alpha\theta} La < 1$;
(3) $2\alpha + Ke^{\alpha\theta} L + N^{-1} \ln(1 + Ke^{\alpha\theta} L) < 0$.

Then system (5.16) *has a unique asymptotically stable solution* $y_0(t) \in \mathcal{AP}(\mathbb{T}_0)$.

8. Conclusion

In our paper, the connection between a specific type of differential equations on time scales (DETC) with impulsive differential equations is established. Some benefits of the established connection include knowledge about properties of linear DETC, the investigation of the existence of periodic and almost periodic solutions, and their stability. We suppose that the problems of stability, oscillations, smoothness of solutions, integral manifolds, theory of functional differential equations, etc. can be investigated by applying our results. Another interesting opportunity is to analyze equations with more sophisticated time scales.

Acknowledgement

The authors wish to express their sincere gratitude to the referee for valuable suggestions which helped to improve the paper.

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