On Asymptotic Equivalence of Impulsive Linear Homogeneous Differential Systems

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Abstract

Sufficient conditions of asymptotic equivalence of impulsive linear homogeneous differential equations are obtained. A partial result is also proved when one of the equations is of delay type.

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1 Introduction

The problem of asymptotic equivalence for linear and non-linear ordinary differential equations have been considered by many authors [1]- [8] (see also bibliography of [1, 3]). And recently, there has been a lot of activity with solutions which undertake discontinuities or jumps at some definite instants [9]. It generates needes 0f investigation the asymptotic behavior of impulsive systems [10]. Interesting article which motivate our study here in this article were written by M. Ráb [7]. Following the results of M. Ráb without requiring any special form and any boundedness condition for the solutions new results for asymptotic equivalence of impulsive systems are obtained. We should also note that Ráb did not consider asymptotic equivalance but rather gave aymptotic representation of solutions.

Let Z, R be sets of all integers and real numbers respectively, $R_+ = [t_0, \infty)$ for some $t_0 \in R$, $||\cdot||$ be the euclidean norm in $R^n, n \in N$.

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We consider the following systems of impulsive differential equations

$$dx/dt = A(t)x, \quad t \neq \theta_i,$$
$$\Delta x(\theta_i) = B_i x(\theta_i) \tag{1}$$

and

$$dy/dt = [A(t) + C(t)]y, \quad t \neq \theta_i,$$
$$\Delta y(\theta_i) = [B_i + D_i]y(\theta_i) \tag{2}$$

where $x, y \in R^n$,

(C_1) $A(t), C(t), t \in B_i, D_i, i \in Z$, are real valued $n \times n-$ matrices, $A(t), C(t) \in C(R_+)$;

(C_2) $\{\theta_i\} \subset R, t_0 < \theta_1 < \theta_2 < \ldots \theta_i \to \infty$ as $i \to \infty$;

(C_3) matrices B_i, D_i satisfy the inequalities

$$\det(I + B_i) \neq 0, \det(I + B_i + D_i) \neq 0, \text{for all} i \in Z. \tag{3}$$

Remark 1.1 The conditions $(C_1) - (C_3)$ imply [9] (p.44) that solutions $x(t, t_0, x_0)$ and $y(t, t_0, x_0)$ of Cauchy's problem for systems (1) and (2) with any $x_0, y_0 \in R^n, t_0 \in R$ exist and unique on R_+.

Definition 1 *[3] A one-to-one correspondence between solutions $x(t)$ of (1) and solutions $y(t)$ of (2) is called an asymptotic equivalence if $x(t) - y(t) \to 0$ as $t \to \infty$.*

The technique used in the present paper is also applied in a certain sense when (2) is replaced by a impulsive delay differential equation of the form

$$dy/dt = A(t)y(t) + C(t)y(t - \tau)], \quad t \neq \theta_i,$$
$$\Delta y(\theta_i) = B_i y(\theta_i) + D_i y(\theta_{i-p}), \tag{4}$$

where τ, p are positive real and integer numbers respectively.

2 Ordinary Differential Equations

Let $X(t)$ be [9] $(p.47)$ a fundamental matrix solutionof (1). We start with the change of dependent variable

$$y(t) = X(t)u(t) \tag{5}$$

which transforms (2) into the system

$$du/dt = P(t)u, \;\; t \neq \theta_i,$$
$$\Delta u(\zeta_i) = Q_i u(\zeta_i) \tag{6}$$

where

$$P(t) = X^{-1}(t)B(t)X(t), Q_i = X^{-1}(\theta_i+)D_i X(\theta_i). \tag{7}$$

The substitution appearing in (5), which was also employed in [7, 6], is very crucial in the proof of our results. In fact, our method is based on a complete characterization of the function $u(t)$ for t sufficiently large.

Let us assume that

$$\int_{t_0}^{\infty} || \, P(t) \, || \, dt + \sum_{t_0<\theta_i<\infty} || \, Q_i \, || < \infty \tag{8}$$

and construct via successive approximations a sequence $\{\Psi_k(t)\}$ of $n \times n$ matrices defined for $t \in [t_0, \infty)$ as follows:

$$\Psi_k(t) = -\int_t^{\infty} P(s)\Psi_{k-1}(s)\, ds - \sum_{t \leq \theta_i < \infty} Q_i \Psi_{k-1}(\theta_i) \quad k = 1, 2, \ldots,$$
$$\Psi_0(t) = I,$$

where I denotes the $n \times n$ identity matrix and t_0 is a positive real number.

It is clear from (8) that for a given ϵ, $0 < \epsilon < 1$, there exists a $t_1(\epsilon) \geq t_0$ such that

$$\int_t^{\infty} || \, P(s) \, || \;\; ds + \sum_{t<\theta_i<\infty} || \, Q_i \, || < \epsilon \quad \text{for all } t > t_1.$$

It follows that if $t \geq t_1$ then $|| \, \Psi_k(t) \, || < \epsilon^k$ for $k = 1, 2, \ldots$ and hence, by using the Weierstrass M-test, the series $\sum_{k=1}^{\infty} \Psi_k(t)$ converges uniformly for $t \in [t_1, +\infty)$ to the piecewise-continuous function.

We define

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t), \quad t \geq t_1, \tag{9}$$

and note that $\Psi(t)$ satisfies

$$\Psi(t) = -\int_t^{\infty} P(s)[I + \Psi(s)]\, ds - \sum_{t<\theta_i<\infty} Q_i[I + \Psi(\theta_i)] \quad . \tag{10}$$

Theorem 1 *Let $X(t)$ be a fundamental matrix solution of (1) and let $P(t)$ and $\Psi(t)$ be the matrices given by (7) and (9), respectively. If (8) is satisfied and*

$$\lim_{t \to \infty} X(t)\Psi(t) = 0 \tag{11}$$

then (1) and (2) are asymptotically equivalent.

Proof. Let t be sufficiently large, $t \geq t_1$ say. Then in view of (10) we see that the function $u(t) = [I + \Psi(t)]c$, $c \in R^n$, is a solution of (6) defined on $[t_1, \infty)$ and hence $y(t) = X(t)[I + \Psi(t)]c$ is a solution of (2) .

Let $x^0 = X(t_2)c$ and $y^0 = X(t_2)(I+\Psi(t_2))c$. Denote by $x(t, c) = x(t, t_2, x^0)$ and $y(t, c) = y(t, t_2, y^0)$ the solutions of (1) and (2) satifying $x(t_2) = x^0$ and $y(t_2) = y^0$, respectively.

On the other hand, since $\Psi(t) \to 0$ as $t \to \infty$, we note that there exists a $t_2 > t_1$ such that $I+\Psi(t_2)$ is nonsingular.

Now, because of the existence and uniqueness of solutions of linear differential equations and the fact that $I + \Psi(t_2)$ is nonsingular, the relation $y^0 = X(t_2)[I + \Psi(t_2)]X^{-1}(t_2)x^0$ defines a one-to-one correspondence between solutions $x(t)$ of (1) and $y(t)$ of (2) such that $y(t) = x(t) + X(t)\Psi(t)c$ for $t > t_1$. This, in view of (11), completes the proof.

One can see that if impulsive parts in (1) and (2) are cancelled then it is easy to obtain a corresponding result for ordinary differential equations.

3 Delay Differential Equation

We use the substitution $y(t) = X(t)u(t)$ to transform (4) into the system

$$\begin{aligned} &u'(t) = Q(t)u(t - \tau), \ \ t \neq \theta_i, \\ &\Delta u(\theta_i) = W_i u(\theta_{i-p}), \end{aligned} \tag{12}$$

where

$$Q(t) = X^{-1}(t)B(t)X(t - \tau), W_i = X^{-1}(\theta_i+)D_i X(\theta_{i-p}). \tag{13}$$

We construct a sequence $\{\Phi_k\}$ of $n \times n$ matrices as follows:

$$\begin{aligned} &\Phi_k(t) = -\int_t^{\infty} Q(s)\Phi_{k-1}(s - \tau)\, ds - \sum_{t \leq \theta_i < \infty} W_i \Phi_{k-1}(\theta_{i-p}) \quad k = 1, 2, \ldots, \\ &\Phi_0(t) = I, \end{aligned}$$

It can be shown similarly as in the previous section that if

$$\int_{t_0}^{\infty} || \, Q(t) \, || \, dt + \sum_{t_0 \leq \theta_i < \infty} || \, W_i \, || < \infty \tag{14}$$

then $\sum_{k=1}^{\infty} \Phi_k(t)$ converges uniformly for $t \in [t_1, \infty)$, where $t_1 \geq t_0$ is a sufficiently large real number. As in the previous section, it follows that the matrix function

$$\Phi(t) = \sum_{k=1}^{\infty} \Phi_k(t), \quad t \geq t_1. \tag{15}$$

satisfies

$$\Phi(t) = -\int_t^\infty Q(s)[I + \Phi(s-\tau)]\,ds + \sum_{t \le \theta_i < \infty} W_i[I + \Phi(\theta_{i-p})], \quad t \ge t_1,$$

and therefore the fuction $u(t) = [I + \Phi(t)]c$, $c \in R^n$, is a solution of (12).

Unfortunately, we cannot prove a one-to-one correspondence between solutions of (1) and (4). However, it is possible to give a similar relation between the set of solutions of (1) and a subset of solutions of (4).

We give without proof the following two theorems as they can be proved in a similar fashion.

Theorem 2 *Let $X(t)$ be a fundamental matrix solution of (1) and let $Q(t), W_i$ and $\Phi(t)$ be the matrices given by (13) and (15), respectively. If (14) holds, then $y(t) = X(t)[I + \Phi(t)]c$ is a solution of (4) for t sufficienly large.*

Theorem 3 *Let $X(t)$ be a fundamental matrix solution of (1) and let $Q(t), W_i$ and $\Phi(t)$ be the matrices given by (13) and (15), respectively. If (14) is satisfied and*

$$\lim_{t \to \infty} X(t)\Phi(t) = 0$$

then for a given solution $x(t)$ of (1) there is a solution $y(t)$ of (4) such that $x(t) - y(t) \to 0$ as $t \to \infty$.

It is easy to observe that the above arguments applies if (4) is replaced by a more general linear functional differential equation.

4 Example

We provide an example to illustrate a result of this paper. Consider the systems

$$x'(t) = ax, \ \ t \ne i, \quad \Delta x(i) = mx(i), \tag{16}$$

and

$$y'(t) = [a + s(t)]y, \ \ t \ne i, \quad \Delta y(i) = [m + q^i]y(i), \tag{17}$$

where $a, m \in R, i \in Z$, $s(t) \in C([t_0, \infty))$ for some $1 > t_0 > 0$. We assume that there exist real numbers $K_0 > 0$ and $b < 0$ such that

$$|\, s(t) \,| < K_0 e^{b\,t} \quad \text{for all } t \ge t_0, \tag{18}$$

and $q \in R$, satisfies $|q| < 1$. And therefore

$$\int_{t_0}^\infty |s(t)|\,dt + \sum_{t_0 \le i < \infty} |q|^i < K_0 \frac{1}{|b|} e^{bt_0} + \frac{|q|}{1 - |q|} < \infty.$$

Let $i(t, t_0)$ be a number of points i in the interval (t, t_0). Then [9] $X(t) = e^{a(t-t_0)}(1+m)^{i(t,t_0)}$ is a solution of (16) such that $X(t_0) = 1$ and there exists $K \in R, K > 0$ such that

$$\mid X(t) \mid \leq K e^{(a+\ln(1+m))t} \quad \text{for all } t \geq t_0,$$

Denote $\alpha = \max\{b, \ln q\}, K_n = \frac{1}{n|\alpha|} + \frac{1}{1-e^{n\alpha}}$. One can show that $\mid \Psi_n(t) \mid \leq K_1 \dots K_n e^{n\alpha t}$, and

$$\mid \Psi(t) \mid \leq \Pi_{n=1}^{\infty} K_1 \dots K_n e^{n\alpha t} \leq \Pi_{n=1}^{\infty} K_1^n e^{n\alpha t} \leq e^{(\ln K_1 + \alpha)t} \frac{1}{1 - e^{\alpha t}}.$$

Hence, if $a + \ln(1+m) + \ln K_1 + \alpha < 0$ then $X(t)\Psi(t) \to 0$ as $t \to \infty$. Thus all conditions of Theorem 1 are fulfilled and we may conclude that systems (16) and (17) are asymptotically equivalent.

References

[1] **Cesari L.** Asymptotic behavior and stability problems in ordinary differential equations, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.

[2] **Levinson N.** // Duke Math. J. 1948. Vol. 15, 111-126.

[3] **Nemytskii V.V. and Stepanov V.V.** Qualitative theory of Differential Equations, Princeton, New Jersey, Princeton University Press, 1966.

[4] **Saito S.**// Math. Japan, 1992. Vol. 37, 503-513.

[5] **Švec M.** // Czech. Math. J. 1974. Vol. 29 , 44-58.

[6] **Wintner A.** // Amer. Jour. Math. 1946. Vol. 68, 185-213.

[7] **Ráb M.** // Czech. Math. J. 1958. Vol. 83, 222-229.

[8] **Yakubovich V.A.** // Mat. Sbornik 1951. Vol. 28, 217-240.

[9] **Samoilenko A. M. and Perestyuk N. A.** Impulsive Differential Equations, World Scientific, Singapore, 1995.

[10] **Simeonov P.S. and Bainov D.D.** // J. Math. Anal. Applic. 1998. Vol. 135, 591-610.