PERGAMON

Nonlinear Analysis 48 (2002) 271–286

Nonlinear Analysis

www.elsevier.com/locate/na

The control of boundary value problems for quasilinear impulsive integro-differential equations[☆]

M.U. Akhmetov[a,1], A. Zafer[a,*], R.D. Sejilova[b]

[a] Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey
[b] Aktobe Higher Military Aviation Academy, 463019 Aktobe, Kazakhstan

Received 10 March 1999; accepted 29 September 1999

Keywords: Impulse; Control; Integro-differential equation; Quasilinear system

1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is richer in problems in comparison with the corresponding theory of differential equations and many of the mathematical problems encountered in studying the impulsive differential equations cannot be treated with the usual techniques of ordinary differential equations [7,9,11]. Moreover, impulsive differential systems represent a natural framework for mathematical modelling of several processes in natural science. That is why in recent years the study of such systems has been very intensive. However, the theory of integro-differential equations with impulse actions on surfaces has not yet been sufficiently elaborated when compared to that of impulsive differential equations and integro-differential equations. There are also several problems on the controllability problem for impulsive systems which are connected with the results of the theory of integral and integro-differential equations and have not been well investigated yet [5,6,8,10].

☆ The first and third authors were partially supported by INTAS under the Grant 96-0915.

* Corresponding author. Tel.: +312-210-2995; fax: +312-210-1282.
E-mail address: zafer@metu.edu.tr (A. Zafer).

[1] On leave from the West Kazakhstan Economical and Financial Institute, 463000 Aktobe, Kazakhstan.

PII: S0362-546X(00)00186-3

In this paper we are concerned with the controllability problem of solutions of integro-differential equations of the form

$$dx/dt = A(t)x + \int_{\alpha}^{t} K(t,s)x(s)\,ds + C(t)u(t) + f(t) + \mu g(t,x,u,\mu), \quad t \neq \theta_i + \mu\tau_i(x,\mu),$$

$$\Delta x(\zeta_i) = B_i x(\zeta_i) + \sum_{\alpha<\zeta_j\le\zeta_i} D_{ij}x(\zeta_j) + Q_i v_i + J_i + \mu W_i(x(\zeta_i), v_i, \mu),\ i = 1,2,\ldots,p, \tag{1}$$

$$x(\alpha) = a, \quad x(\beta) = b, \tag{2}$$

where $\mu > 0$ is a small parameter, $\alpha, \beta, \theta_i, \zeta_i \in R$ such that $\alpha < \theta_1 < \cdots < \theta_p < \beta$, and $\zeta_i = \theta_i + \mu\tau_i(x(\zeta_i),\mu)$; A, K, D_{ij} and B_i are $n \times n$ matrices; C and Q_i are $n \times m$ matrices; x, f, g, J_i, W_i, a, and b are n-vectors; u and v are m-vectors; $\Delta x(t)$ denotes $x(t+)-x(t-)$, where $x(t+)=\lim_{h\to 0^+}x(t+h)$ and $x(t-)=\lim_{h\to 0^-}x(t+h)$. We assume that solutions are left continuous and therefore write $\Delta x(t) = x(t+) - x(t)$.

The objective of this paper is to find sufficient conditions for controllability of the boundary value problems for quasilinear impulsive system of integro-differential equations of the form (1), (2). We obtain our results by comparing solutions of integro-differential equations having impulse actions at variable moments with solutions of integro-differential equations having impulse actions at fixed moments. The results of this paper may be considered as a continuation and a generalization of the results obtained in [3], where the control of boundary value problems for quasilinear impulsive differential systems was considered. Moreover, for our purpose we have also proved theorems on existence and uniqueness of solutions of integro-differential equations with impulse actions at fixed moments and obtained the integral representation of solutions of such systems.

2. Preliminaries

In what follows we denote by $PAC[\alpha,\beta]$ the set of all functions $x : [\alpha,\beta] \to R^n$ piecewise absolutely continuous and continuous on the left with discontinuities of the first kind at points θ_i, $i=1,2,\ldots,p$, by $L_2^r[\alpha,\beta]$ the set of all square integrable functions $\phi : [\alpha,\beta] \to R^r$ and by $D^r[1,p]$ the set of all finite sequences $\{\xi_i\}, \xi_i \in R^r$, $i=1,\ldots,p$, where p and r are fixed positive integers. Furthermore, we define $\Pi^r[\alpha,\beta] = L_2^r[\alpha,\beta] \times D^r[1,p]$ and identify its elements as $\{\phi,\xi\}$, and let

$$\langle\{\phi,\xi\},\{\omega,v\}\rangle = \int_{\alpha}^{\beta} (\phi,\omega)\,dt + \sum_{i=1}^{p}(\xi_i, v_i)$$

be an inner product in $\Pi^r[\alpha,\beta]$, where $(,)$ is the Euclidean scalar product in R^r.

Throughout this paper we need the following conditions:

(c_1) the functions g, W_i, τ_i, $i=1,2,\ldots,p$, are continuous with respect to their variables and continuously differentiable with respect to x, u, and v;

(c_2) the elements of matrix $K(t,s) : [\alpha,\beta]\times[\alpha,\beta] \to R^n \times R^n$ are square integrable with respect to s;
(c_3) the columns of matrix $A(t)$ and the function $f(t) : [\alpha,\beta] \to R^n$ are in $L_2^n[\alpha,\beta]$;
(c_4) $\{f,J\} \in \Pi^n[\alpha,\beta]$;
(c_5) $\det(I + B_j + D_{jj}) \neq 0$, $\det(I + B_j) \neq 0$ for $j = 1,2,\dots,p$.

The process defined by (1) for fixed μ and $\{u,v\}$ operates as follows: the point $P_t(t,x(t))$, starting at (α,a), moves along the curve defined by the solution $x(t) = x(t,\alpha,a)$ of the equation

$$\frac{dx}{dt} = A(t)x(t) + \int_\alpha^t K(t,s)x(s)\,ds + C(t)u + f(t) + \mu g(t,x,u,\mu). \tag{3}$$

The motion along this curve terminates at time $t = \zeta_1$ when the point P_t arrives at the surfaces of discontinuity so that $\zeta_1 = \theta_1 + \mu\tau_1(x(\zeta_1),\mu)$. At that moment the point P_t performs a jump

$$\Delta x|_{t=\zeta_1} = B_1x(\zeta_1) + D_{11}x(\zeta_1) + Q_1v_1 + J_1 + \mu W_1(x(\zeta_1),v_1,\mu)$$

and proceeds to move along the curve described by the solution $x(t,\zeta_1,x(\zeta_1+))$ of system (3), until it meets the next surface of discontinuity, and so on. We should note that each solution of (1) is a function from $PAC[\alpha,\beta]$ provided it is defined on $[\alpha,\beta]$.

If for any given bounded set $G \subset R^n$ there exists a positive real number μ_0, $\mu_0 = \mu_0(G)$, such that for all arbitrary $a,b \in G$ and $\mu < \mu_0$ there is a control $\{u,v\} \in \Pi^m$ for which system (1) admits a solution $x(t)$ satisfying (2), then the boundary value problem (1), (2), which we shall denote by Σ_μ, is said to be solvable.

We should note that system (1) considered in this paper belongs to a class of systems with impulses at non-fixed moments. It is possible for the integral curve of such system to meet more than one time and even infinitely many times one and the same surface of discontinuity. This phenomenon is called beating [9,11]. Therefore, the investigation of such a system needs conditions for the absence of beating.

Let s be a positive real number, and let Π_s be the subspace of elements (x,u,v) satisfying the inequality $|x| + |u| + |v| \le s$, where $|\cdot|$ is the Euclidean norm in R^n.

For fixed positive real numbers H and μ_1, define

$$G_H = \{(x,u,v,t,i,\mu) : (x,u,v) \in \Pi_H, \quad \alpha \le t \le \beta,\ i = 1,2,\dots,p,\ \mu \le \mu_1\}.$$

If we let

$$m_1 = \max\left\{\sup_t |A(t)|, \sup_t |C(t)|, \sup_{t,s} |K(t,s)|, \max_i |B_i|, \max_{ij} |D_{ij}|\right\},$$
$$m_2 = \max\left\{\sup_t |f(t)|, \max_i |J_i|\right\},$$
$$m_3 = \max\left\{\max_{G_H} |g|, \max_{G_H} |W|, \max_{G_H} |\tau|\right\}$$

then it is not difficult to prove that the following lemma holds.

Lemma 1. *If* $\mu_1 m_3 < \min\{\theta_1 - \alpha, \beta - \theta_p\}$, *and if for any* $x, \mu, i \in G_H$, $\theta_{i+1} + \mu\tau_{i+1}(x,\mu) < \theta_i + \mu\tau_i(x,\mu)$, *then every solution of* (1) *which is in* G_H *and defined on* $[\alpha, \beta]$ *intersects each surface* $t = \theta_i + \mu\tau_i(x,\mu)$, $i = 1, 2, \ldots, p$, *at least once.*

To obtain conditions for the absence of beating we first observe in view of (c_1) that there is a positive real number L such that uniformly in G_H,

$$|g(t,x_1,u_1,v^1,\mu) - g(t,x_2,u_2,v^2,\mu)| \le L\left\{|x_1 - x_2| + |u_1 - u_2| + |v^1 - v^2|\right\},$$

$$|W_i(x_1,v^1,\mu) - W_i(x_2,v^2,\mu)| \le L\left\{|x_1 - x_2| + |v^1 - v^2|\right\},$$

$$|\tau_i(x_1,\mu) - \tau_i(x_2,\mu)| \le L\,|x_1 - x_2|.$$

If we now define

$$\gamma(\mu) = m_1 H(2 + (\beta - \alpha)) + m_2 + \mu m_3$$

we easily obtain the following result, the proof of which is similar to that of Lemma 5 on p. 22 in [11] and hence is omitted.

Lemma 2. *Let system* (1) *be considered in* G_H, *and let* $\mu < \mu_2$, *where* $\mu_2 \le \mu_1$ *is a positive real number such that* $\mu_2 L\gamma(\mu_2) < 1$. *If*

$$\tau_i(x,\mu) \ge \tau_i\left((I + B_i + D_{ii})x + \sum_{j=1}^{i-1} D_{ij}x^{(j)} + Q_i v_i + J_i + \mu W_i(x,v,\mu), \mu\right)$$

for $|x| < H$, $|x^{(j)}| < H$, $j = 1, 2, \ldots, i-1$, *and* $|v| < H$, *then every solution* $x(t)$ *of* (1) *meets any given surface* $t = \theta_i + \mu\tau_i(x,\mu)$ *at most once.*

We will also need the following lemmas in the proof of our results. These two lemmas are analogous to the Fubini's theorem on changing the order of integration and may therefore find further applications in other branches of mathematics as well.

Lemma 3. *Let* D_{ij}, $i, j = 1, 2, \ldots, p$, *be constant matrices of size* $n \times n$ *and* $\{\xi_i\} \in D^n[1, p]$. *Then*

$$\sum_{\alpha<\theta_i<t}\ \sum_{\alpha<\theta_j\le\theta_i} D_{ij}\xi_j = \sum_{\alpha<\theta_i<t}\ \sum_{\theta_i\le\theta_j<t} D_{ji}\xi_i \quad \textit{for each } t \in [\alpha, \beta].$$

Proof. The lemma can be proved simply by grouping the terms. □

Lemma 4. *Let* $K(t,s)$ *be a matrix of size* $n \times n$. *If* $K(t,s)$ *is square integrable with respect to* s *on* $[\alpha, \beta]$ *for each fixed* $t \in [\alpha, \beta]$, *and* $\phi_i(t) \in L_2^n[\alpha, \beta]$ *for* $i = 1, 2, \ldots, n$, *then*

$$\int_\alpha^t K(t,s) \sum_{\alpha<\theta_i<s} \phi_i(s)\,\mathrm{d}s = \sum_{\alpha<\theta_i<t} \int_{\theta_i}^t K(t,s)\phi_i(s)\,\mathrm{d}s \quad \textit{for each } t \in [\alpha, \beta]. \qquad (4)$$

Proof. Define $\Phi_i(t)$ for $i=1,2,\ldots,p$ as follows: $\Phi_i(t)=0$ if $t\le\theta_i$, and $\Phi_i(t)=\phi_i(t)$ if $t>\theta_i$. Then it follows that

$$\int_\alpha^t K(t,s)\sum_{\alpha<\theta_i<t}\Phi_i(s)\,\mathrm{d}s=\sum_{\alpha<\theta_i<t}\int_{\theta_i}^t K(t,s)\Phi_i(s)\,\mathrm{d}s=\sum_{\alpha<\theta_i<t}\int_\alpha^t K(t,s)\phi_i(s)\,\mathrm{d}s$$

which clearly proves (4). □

3. Existence of solutions of integro-differential equations

We first consider the integral equation

$$x(t)=\int_\alpha^t G(t,s)x(s)\,\mathrm{d}s+\sum_{\alpha<\theta_i<t}M_i(t)x(\theta_i)$$
$$+\sum_{\alpha<\theta_i<t}N_i(t)x(\theta_i+)+\sum_{\alpha<\theta_i<t}J_i+f(t), \tag{5}$$

under the following conditions:

(a_1) the columns of matrix $G(t,s):[\alpha,\beta]\times[\alpha,\beta]\to R^n\times R^n$ are square integrable with respect to s;
(a_2) the columns of matrices $M_i(t)$ and $N_i(t)$ and the function $f(t)$ belong to $PAC[\alpha,\beta]$;
(a_3) $\det(I-N_j(\theta_j+)+N_j(\theta_j))\neq 0$ for $j=1,2,\ldots,p$.

We shall employ a resolvent argument used by Anokhin et al. in [4] in order to write (5) in a more useful form. So, let $R(t,s)$ be the resolvent of the Volterra integral equation with kernel $G(t,s)$. In view of Lemma 4, it is easy to show that (5) is equivalent to

$$x(t)=\sum_{\alpha<\theta_i<t}\left[\int_{\theta_i}^t R(t,s)M_i(s)\,\mathrm{d}s+M_i(t)\right]x(\theta_i)$$
$$+\sum_{\alpha<\theta_i<t}\left[\int_{\theta_i}^t R(t,s)N_i(s)\,\mathrm{d}s+N_i(t)\right]x(\theta_i+)$$
$$+\sum_{\alpha<\theta_i<t}\left[\int_{\theta_i}^t R(t,s)\,\mathrm{d}s+I\right]J_i+\int_\alpha^t R(t,s)f(s)\,\mathrm{d}s+f(t), \tag{6}$$

where I is the $n\times n$ identity matrix.

Introducing the notations

$$M_{ij}=\int_{\theta_i}^{\theta_j} R(\theta_j,s)M_i(s)\,\mathrm{d}s+M_i(\theta_j),$$

$$N_{ij}=\int_{\theta_i}^{\theta_j} R(\theta_j,s)N_i(s)\,\mathrm{d}s+N_i(\theta_j),$$

$$p_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j, s)\,\mathrm{d}s + I,$$

we obtain from (6) that

$$x(\theta_j) = \sum_{\alpha<\theta_i<\theta_j} \left[M_{ij}x(\theta_i) + N_{ij}x(\theta_i+)\right] + \sum_{\alpha<\theta_i<\theta_j} p_{ij}J_i + \int_{\alpha}^{\theta_j} R(\theta_j, s)f(s)\,\mathrm{d}s + f(\theta_j). \tag{7}$$

In view of (7) we also obtain from (6) that

$$\begin{aligned} x(\theta_j+) = {} & [I - N_j(\theta_j+) + N_j(\theta_j)]^{-1}\{(I + M_j(\theta_j+) - M_j(\theta_j))x(\theta_j) \\ & + \sum_{\alpha<\theta_i<\theta_j} [(M_i(\theta_j+) - M_i(\theta_j))x(\theta_i) \\ & + (N_i(\theta_j+) - N_i(\theta_j))x(\theta_i+)] + f(\theta_j+) - f(\theta_j) + J_i\}. \end{aligned} \tag{8}$$

Expressions (7) and (8) recursively determine $x(\theta_j)$ and $x(\theta_j+)$. Since the non-homogeneous part of this system is a linear combination of $\int_{\alpha}^{\theta_i} R(t,s)f(s)\,\mathrm{d}s$, $f(\theta_i)$, and J_i, for $i = 1, 2, \ldots, p$, it follows that $x(\theta_j)$ and $x(\theta_j+)$ can be written as linear combinations of these vectors with certain matrix coefficients. Thus we obtain that the following theorem is true.

Theorem 1. *Let conditions* (a_1)–(a_3) *hold. Then system* (5) *has a unique solution* $x(t) \in PAC[\alpha, \beta]$ *which can be represented as*

$$x(t) = \int_{\alpha}^{t} P_1(t,s)f(s)\,\mathrm{d}s + \sum_{\alpha<\theta_i<t} Q_i(t)J_i + \sum_{\alpha<\theta_i<t} P_2^i(t)f(\theta_i) + f(t) + \sum_{\alpha<\theta_i<t} J_i,$$

where $Q_i(t)$, $P_2^i(t)$, $i = 1, 2, \ldots, p$, *and* $P_1(t,s)$ *are certain piecewise continuous function matrices of size* $n \times n$.

We may now state and prove the theorem on existence and uniqueness of solutions of the impulsive system of integro-differential equations of the form

$$\begin{aligned} & \mathrm{d}x/\mathrm{d}t = A(t)x + \int_{\alpha}^{t} K(t,s)x(s)\,\mathrm{d}s + f(t), \quad t \neq \theta_i, \\ & \Delta x(\theta_i) = B_i x(\theta_i) + \sum_{\alpha<\theta_j\leq\theta_i} D_{ij}x(\theta_j) + J_i. \end{aligned} \tag{9}$$

For some related results we refer in particular to [1].

Theorem 2. *Let conditions* (c_2)–(c_4) *be satisfied. Then for a given* $x_0 \in R^n$ *there exists a unique solution* $x(t) \in PAC[\alpha, \beta]$ *of* (9) *which satisfies* $x(\alpha) = x_0$.

Proof. It is easy to verify that the integro-differential equation

$$x(t)=x_0+\int_\alpha^t A(s)x(s)\,\mathrm{d}s+\int_\alpha^t\int_\alpha^\sigma K(\sigma,s)x(s)\,\mathrm{d}s\,\mathrm{d}\sigma+\sum_{\alpha<\theta_i<t}B_ix(\theta_i)$$
$$+\int_\alpha^t f(s)\,\mathrm{d}s+\sum_{\alpha<\theta_i<t}\sum_{\alpha<\theta_j\le\theta_i}D_{ij}x(\theta_i)+\sum_{\alpha<\theta_i<t}J_i \tag{10}$$

is equivalent to (9) with $x(\alpha)=x_0$.

By letting

$$\Psi(t,s)=A(s)+\int_s^t K(\sigma,s)\,\mathrm{d}\sigma,$$
$$\Phi_i=B_i+\sum_{\theta_i\le\theta_j<t}D_{ji},$$
$$F(t)=x_0+\int_\alpha^t f(s)\ \mathrm{d}s,$$

and using the Fubini's theorem together with Lemma 3, it follows from (10) that

$$x(t)=\int_\alpha^t \Psi(t,s)x(s)\,\mathrm{d}s\,\mathrm{d}\sigma+\sum_{\alpha<\theta_i<t}\Phi_ix(\theta_i)+\sum_{\alpha<\theta_i<t}J_i+F(t). \tag{11}$$

In view of Theorem 1 we may conclude that (11) has a unique solution $x(t)\in PAC[\alpha,\beta]$. This completes the proof. □

Next, we consider the impulsive integro-differential equation

$$\partial\lambda(t,s)/\partial s=-\lambda(t,s)A(s)-\int_s^t\lambda(t,\sigma)K(\sigma,s)\,\mathrm{d}\sigma,\quad s\ne\theta_i,\quad t\in[\alpha,\beta],$$
$$\Delta\lambda(t,\theta_i)=-\lambda(t,\theta_i)B_i(I+B_i)^{-1}-\sum_{\theta_i\le\theta_j<t}\lambda(t,\theta_j+)D_{ji}(I+B_i)^{-1}, \tag{12}$$

where $\lambda\in R^n$ is a row vector, A, K, D_{ij}, and B_i are as before, and $\Delta\lambda(t,\theta_i)\equiv\lambda(t,\theta_i+)-\lambda(t,\theta_i)$.

Theorem 3. *Let conditions* (c_2)–(c_5) *hold. Then for a given* $h_0\in R^n$ *the system* (12) *has a unique solution* $\lambda(t,s)$ *such that* $\lambda(t,t)=\lambda_0$.

Proof. System (12) is equivalent to the integro-differential equation

$$\lambda(t,s)=\lambda(t,t)-\int_s^t\left[\lambda(t,\sigma)A(\sigma)+\int_\sigma^t\lambda(t,v)K(v,\sigma)\,\mathrm{d}v\right]\,\mathrm{d}\sigma$$
$$-\sum_{s\le\theta_i<t}\lambda(t,\theta_i)B_i(I+B_i)^{-1}-\sum_{s\le\theta_i<t}\sum_{\theta_i\le\theta_j<t}\lambda(t,\theta_j+)D_{ji}(I+B_i)^{-1}. \tag{13}$$

Using slightly different versions of Lemmas 3 and 4 in (13) we get

$$\lambda(t,s) = \lambda_0 - \int_s^t \lambda(t,\sigma)\left[A(\sigma) + \int_s^t K(\sigma,v)\,dv\right] d\sigma$$
$$+ \sum_{s\le\theta_i<t} \lambda(t,\theta_i)B_i(I+B_i)^{-1} - \sum_{s\le\theta_i<t} \lambda(t,\theta_i+) \sum_{s\le\theta_j\le\theta_i} D_{ij}(I+B_i)^{-1}. \quad (14)$$

Comparing (5) and (14) it is not difficult to see that the arguments developed in the proof of Theorem 1 can be used, and therefore one can conclude that system (12) has a unique solution $\lambda(t,s)$ satisfying $\lambda(t,t) = \lambda_0$. □

Now for a fixed i, let $\lambda_i(t,s)$ be the unique solution of (12) satisfying $\lambda_i(t,t) = \delta_{ij}$, where δ_{ij} denotes the Kronecker's delta.

If we define

$$\Lambda(t,s) = \mathrm{col}(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n), \quad \Lambda(t,t) = I,$$

then the following theorem is obtained.

Theorem 4. *Let $x(t) = x(t,\alpha,x_0)$ be a solution of the Cauchy problem for* (9). *Then $x(t)$ has the representation*

$$x(t) = \Lambda(t,\alpha)x_0 + \int_\alpha^t \Lambda(t,s)f(s)\,ds + \sum_{\alpha<\theta_i<t} \Lambda(t,\theta_j+)J_i. \quad (15)$$

Proof. Let $x(t) = x(t,\alpha,x_0)$ be the solution of (9) and let $\phi(s) = \Lambda(t,s)x(s)$. Clearly,

$$\phi(t) - \phi(\alpha) = \int_\alpha^t \phi'(s)\,ds + \sum_{\alpha<\theta_i<t} \Delta\phi(\theta_i). \quad (16)$$

Since

$$\Delta\phi(\theta_i) = \Lambda(t,\theta_i+)x(\theta_i+) - \Lambda(t,\theta_i)x(\theta_i) = \Lambda(t,\theta_i)\Delta x(\theta_i) + \Delta\Lambda(t,\theta_i)x(\theta_i+),$$

we have

$$\sum_{\alpha<\theta_i<t} \Delta\phi(\theta_i) = \sum_{\alpha<\theta_i<t} [\Delta\Lambda(t,\theta_i)(I+B_i) + \Lambda(t,\theta_i)B_i]\,x(\theta_i)$$
$$+ \sum_{\alpha<\theta_i<t}\sum_{\alpha<\theta_j\le\theta_i} \Lambda(t,\theta_i+)D_{ij}x(\theta_j) + \sum_{\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i$$
$$= \sum_{\alpha<\theta_i<t}\left[\Delta\Lambda(t,\theta_i)(I+B_i) + \Lambda(t,\theta_i)B_i + \sum_{\theta_i\le\theta_j<t} \Lambda(t,\theta_j+)D_{ji}\right] x(\theta_i)$$
$$+ \sum_{\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i. \quad (17)$$

In view of the impulse condition in (12), we obtain from (17) that

$$\sum_{\alpha<\theta_i<t} \Delta\phi(\theta_i) = \sum_{\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i. \tag{18}$$

On the other hand, by differentiating the relation $\phi(s) = \Lambda(t,s)x(s)$ we have

$$\phi'(s) = \partial\Lambda/\partial s\, x(s) + \Lambda(t,s)\left[A(s)x(s) + \int_\alpha^t K(s,v)x(v)\,\mathrm{d}v + f(s)\right]$$

and therefore by the help of Fubini's theorem it follows that

$$\begin{aligned}\int_\alpha^t \phi'(s)\,\mathrm{d}s &= \int_\alpha^t \left\{[\partial\Lambda/\partial s + \Lambda(t,s)A(s)]x(s) + \Lambda(t,s)\int_\alpha^t K(s,v)x(v)\,\mathrm{d}v\right\}\mathrm{d}s \\ &\quad + \int_\alpha^t \Lambda(t,s)f(s)\,\mathrm{d}s \\ &= \int_\alpha^t \left[\partial\Lambda/\partial s + \Lambda(t,s)A(s) + \int_\alpha^t \Lambda(t,v)K(v,s)\,\mathrm{d}v\right]x(s)\,\mathrm{d}s \\ &\quad + \int_\alpha^t \Lambda(t,s)f(s)\,\mathrm{d}s. \end{aligned} \tag{19}$$

Since $\Lambda(t,s)$ is a solution of (12) it is clear from (19) that

$$\int_\alpha^t \phi'(s)\,\mathrm{d}s = \int_\alpha^t \Lambda(t,v)f(v)\,\mathrm{d}v. \tag{20}$$

By (16), (18), and (20) we may deduce that

$$\phi(t) = \Lambda(t,\alpha)x_0 + \int_\alpha^t \Lambda(t,s)f(s)\,\mathrm{d}s + \sum_{\alpha<\theta_i<t} \Lambda(t,\theta_j+)J_i.$$

This completes the proof. □

4. A related system with impulses at fixed moments

To investigate problem Σ_μ we shall use a comparison method [2]. For this purpose, we need to construct an integro-differential equation with fixed moments of impulse actions which is associated with (1). We propose

$$\begin{aligned}\mathrm{d}y/\mathrm{d}t &= A(t)y + \int_\alpha^t K(t,s)y(s)\,\mathrm{d}s + C(t)u + f(t) + \sum_{\alpha<\theta_i<t} F_i(t,y,u,\mu), \quad t \neq \theta_i, \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{\alpha<\theta_j<\theta_i} D_{ij}y(\theta_j) + Q_i v_i + J_i + S_i(y,u,v_i,\mu), \end{aligned} \tag{21}$$

where F_i and S_i are some functionals to be determined.

Without any loss of generality we may assume that $\tau_i \geq 0$ in G_H.

Definition 1 (Akhmetov and Perestyuk [2]; Akhmetov and Zafer [3]). The systems (1) and (21) are said to be Ω-equivalent or have property Ω in G_H, if for a fixed positive real number $h < H$ and a sufficiently small μ it is true that: given any solution $x(t)$ of (1), $|x(t)| < h$, $t \in [\alpha, \beta]$, there is a solution $y(t)$ of (21), $|y(t)| < H$, such that $x(t) = y(t)$ for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$, and conversely, given any solution $y(t)$ of (21), $|y(t)| < h$, $t \in [\alpha, \beta]$, there is a solution $x(t)$ of (1), $|x(t)| < H$, such that $x(t) = y(t)$ for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$.

We show in this section that it is possible to choose the functionals S_i and F_i in such a way that systems (1) and (21) become Ω-equivalent in G_H.

Fix h and a positive real number $\mu_3 \le \mu_2$ such that $m_3\mu_3\gamma(\mu_3) < H - h$. Let $\phi(t)$ be a solution of (1) satisfying $\phi(\alpha) = a$ and $|\phi(t)| < h$ for $t \in [\alpha, \beta]$. Denote by ζ_i a point of discontinuity of $\phi(t)$ for each fixed i. Fix k, $1 < k \le p$, and suppose that F_i, S_i, $i = 1, 2, \ldots, k-1$, are so determined that the solution $\eta(t)$ of (21) with $\eta(\alpha) = a$ satisfies

(i) $|\eta(t)| < H \quad$ for $t \in [\alpha, \theta_k]$,
(ii) $\eta(t) = \phi(t) \quad$ for $t \in [\alpha, \theta_k] \setminus \bigcup_{i=1}^{k-1} (\theta_i, \zeta_i]$, and
(iii)

$$\int_\alpha^t \left[A(s)\eta(s) + \int_\alpha^s K(s,\sigma)\eta(\sigma)\,\mathrm{d}\sigma + C(s)u + f(s) + \sum_{\alpha < \theta_i < s} F_i(s, \eta, u, \mu) \right] \mathrm{d}s$$

$$= \int_\alpha^t \left[A(s)\phi(s) + \int_\alpha^s K(s,\sigma)\phi(\sigma)\,\mathrm{d}\sigma + C(s)u + f(s) + \mu g(s, \phi(s), u(s), \mu) \right] \mathrm{d}s$$

for $t \in [\alpha, \theta_k]$.

Since $\zeta_k \ge \theta_k$ we have $\eta(\theta_k) = \phi(\theta_k)$. Let

$$\eta(\theta_k+) = (I + B_k)\eta(\theta_k) + \sum_{\alpha < \theta_j \le \theta_k} D_{kj}\eta(\theta_j) + Q_k v_k + J_k + z, \tag{22}$$

where z is to be determined.

Continue the solution $\eta(t)$ for all $t \in [\theta_k, \zeta_k]$ as the solution of the initial value problem

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = F(t), \qquad \xi(\zeta_k) = \phi(\zeta_k+), \tag{23}$$

where

$$F(t) = A(t)\phi(t) + \int_\alpha^t K(t,\sigma)\phi(\sigma)\,\mathrm{d}\sigma + C(t)u + f(t) + \mu g(t, \phi(t), u(t), \mu).$$

It is clear that

$$\eta(t) = \phi(\zeta_k+) + \int_{\zeta_k}^t F(s)\,\mathrm{d}s \quad \text{for } t \in [\theta_k, \zeta_k].$$

Thus,

$$|\eta(t) - \phi(\zeta_k+)| \le \max_{\theta_k \le t \le \zeta_k} \left| \int_{\theta_k}^{t} F(s)\,\mathrm{d}s \right| \le \mu_3 m_3 \gamma(\mu_3) < H - h,$$

and hence $|\eta(t)| < H$ for $t \in [\theta_k, \zeta_k]$.

Moreover, letting $S_k(\eta, u, v_k, \mu) = z$, we see in view of (22) and

$$\eta(\theta_k+) = \phi(\zeta_k+) + \int_{\zeta_k}^{\theta_k} F(s)\,\mathrm{d}s,$$

that

$$S_k(\eta, u, v_k, \mu) = \phi(\zeta_k+) + \int_{\zeta_k}^{\theta_k} F(s)\,\mathrm{d}s - (I + B_k)\eta(\theta_k)$$

$$- \sum_{\alpha < \theta_j \le \theta_k} D_{kj}\eta(\theta_j) - Q_k v_k - J_k.$$

It is also possible to rewrite S_k as

$$S_k(\eta, u, v_k, \mu) = B_k[(\phi(\zeta_k) - \eta(\theta_k)] + \sum_{\alpha < \zeta_j \le \zeta_k} D_{kj}[\phi(\zeta_j) - \eta(\theta_j)]$$

$$+\mu W_k(\phi(\zeta_k), v_k, \mu). \tag{24}$$

Finally, we let

$$F_k(t, \eta, u, \mu)$$

$$= \begin{cases} A(t)[\phi(t) - \eta(t)] + \displaystyle\int_{\alpha}^{t} K(t,s)[\phi(s) - \eta(s)]\mathrm{d}s + \mu g(t, \phi(t), u(t), \mu), & t \in [\theta_k, \zeta_k], \\ 0 & \text{otherwise}. \end{cases} \tag{25}$$

Due to the construction of F_k and S_k it follows that (1) and (21) have property Ω in $[\alpha, \theta_{k+1}]$. Continuing in the same manner we may conclude that if $\mu \le \mu_3$ then (1) and (21) are Ω-equivalent in $[\alpha, \beta]$.

Thus we have proved the following theorem.

Theorem 5. *If μ is sufficiently small, then systems* (1) *and* (2) *are Ω-equivalent in* $[\alpha, \beta]$.

5. Controllability of impulsive integro-differential equations

We first consider the conrollability problem for (21), (2). This problem is denoted by γ_μ.

Define

$$\Psi(t) = \int_{\alpha}^{t} E(s)E^{\mathrm{T}}(s)\,\mathrm{d}s + \sum_{\alpha < \theta_i < t} P_i P_i^{\mathrm{T}},$$

where $E(t) = \Lambda(\beta, t)C(t)$ and $P_i = \Lambda(\beta, \theta_i+)Q_i$.

Lemma 5. *Let* (c_1)–(c_5) *be satisfied. If* $\Psi(\beta)$ *be nonsingular, then* γ_μ *is a solvable problem, and the solution of this problem is the limit of a uniformly convergent sequence obtained by the method of successive approximations.*

Proof. We will prove that γ_μ is solvable with a control $\{u, v\}$ of the following form:

$$u = E^{\mathrm{T}}(t)c + \hat{u}(t), \quad t \in [\alpha, \beta], \qquad v_i = P_i^{\mathrm{T}} c + \hat{v}_i, \quad i = 1, 2, \dots, p, \tag{26}$$

where $c \in R^n$ is a constant vector, and $\{\hat{u}, \hat{v}\} \in \Pi^m[\alpha, \beta]$ is such that it is orthogonal to all columns of $[E, P_i^{\mathrm{T}}]$.

It is clear that problem γ_μ is equivalent to solving

$$y(t) = \Lambda(t, \alpha)a + \int_\alpha^t \Lambda(t, s)\left[C(s)u(s) + f(s) + \sum_{\alpha < \theta_i < s} F_i(s, y, u, \mu)\right] \mathrm{d}s$$

$$+ \sum_{\alpha < \theta_i < t} \Lambda(t, \theta_i+)[Q_i v_i + J_i + S_i(y, u, v_i, \mu)], \quad x(\beta) = b. \tag{27}$$

Substituting (26) into (27) we have

$$c = \Psi^{-1}(\beta)\left\{b - \Lambda(\beta, \alpha)a - \int_\alpha^\beta \Lambda(\beta, s)\left[f(s) + \sum_{\alpha < \theta_i < s} F_i(s, y, u, \mu)\right] \mathrm{d}s\right.$$

$$\left. - \sum_{i=1}^{p} \Lambda(\beta, \theta_i+)[J_i + S_i(y, u, v_i, \mu)]\right\}. \tag{28}$$

If we set

$$K = b - \Lambda(\beta, \alpha)a - \int_\alpha^\beta \Lambda(\beta, s) f(s)\,\mathrm{d}s - \sum_{i=1}^{p} \Lambda(t, \theta_i+)J_i,$$

$$u_0(t) = E^{\mathrm{T}}(t)\Psi(\beta)^{-1}K + \hat{u}(t), \qquad v_i^0 = P_i^{\mathrm{T}}\Psi(\beta)^{-1}K + \hat{v}_i,$$

$$y_0(t) = \Lambda(t, \alpha)a + \int_\alpha^t \Lambda(t, s)[C(s)u_0(s) + f(s)]\mathrm{d}s + \sum_{\alpha < \theta_i < t} \Lambda(t, \theta_i+)[Q_i v_i^0 + J_i],$$

$$\varphi = (y(t), u(t), v_i), \qquad \varphi_0 = (y_0(t), u_0(t), v_i^0),$$

$$\kappa(t, \varphi_i, \mu) = \mu^{-1}\int_\alpha^t \Lambda(t, s) \sum_{\alpha < \theta_i < s} F_i(s, y, u, \mu)\,\mathrm{d}s,$$

$$\psi(t, \varphi, \mu) = \mu^{-1} \sum_{\alpha < \theta_i < t} \Lambda(t, \theta_i+)S_i(y, u, v_i, \mu),$$

then it follows from (27) and (28) that

$$\varphi = \varphi_0 + \mu\mathscr{P}(\varphi, \mu), \tag{29}$$

where $\mathscr{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}^i)$,

$$\mathscr{P}_1(t,\varphi) = \kappa(t,\varphi,\mu) + \psi(t,\varphi,\mu) - \Psi(t)\Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)],$$

$$\mathscr{P}_2(t,\varphi) = E(t)^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)],$$

$$\mathscr{P}^i(\varphi) = P_i^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)].$$

Consider a space $\mathscr{B}$ of functions φ of the form $\varphi = (y(t), u(t), v_i)$ with norm

$$\|\varphi\| = \max_t |y(t)| + \max_t |u(t)| + \max_i |v_i|,$$

where $y(t) \in PAC[\alpha,\beta]$ and $\{u,v\} \in \Pi^m[\alpha,\beta]$.

We may assume that the real number h that is fixed in the previous section also satisfies $\|\varphi_0\| < h$ and there is a positive real number $\mu_4 \le \mu_3$ such that

$$\mu_4 \max_{\substack{\|\phi\|\le h \\ 0<\mu\le\mu_1}} \|\mathscr{P}(\varphi,\mu)\| < h - \|\phi_0\|.$$

Let $\mathscr{X} = \{\varphi\colon \varphi \in \mathscr{B},\ \|\varphi\| \le h\}$. It is easy to see that if $\mu \le \mu_4$ then the operator $\phi_0 + \mu\mathscr{P}(\varphi,\mu)$ maps $\mathscr{X}$ into itself. We will show that if μ is sufficiently small then $\mu\mathscr{P}$ becomes a contraction mapping on $\mathscr{X}$.

We first show that operators κ and ψ satisfy a Lipshitz condition with respect to φ. For this purpose let $\varphi_j = (\eta_j, u_j, v_i^{(j)}) \in \mathscr{X}$ for $j = 1,2$.

Fix k, $1 < k \le p$, and let ϕ_j, $j = 1,2$, be functions defined on $[\alpha, \theta_k]$ such that $\phi_j(t) = \eta_j(t)$ for $t \in [\alpha,\theta_k]\setminus\bigcup_{i=1}^{k-1}\left(\theta_i, \zeta_k^{(j)}\right]$, where $\zeta_k^{(j)}$ is the point of discontinuity of ϕ_j, and that

$$\int_\alpha^t \left[A(s)\phi_j(s) + \int_\alpha^s K(s,\sigma)\phi_j(\sigma)\,\mathrm{d}\sigma + \mu g(s,\phi_j(s),u_j(s),\mu)\right]\mathrm{d}s$$

$$= \int_\alpha^t \left[A(s)\eta_j(s) + \int_\alpha^s K(s,\sigma)\eta_j(\sigma)\,\mathrm{d}\sigma + \sum_{\alpha<\theta_i<s} F_i(s,\eta_j,u_j,\mu)\right]\mathrm{d}s \tag{30}$$

for all $t \in [\alpha,\theta_k]$.

Assume that

$$\|\phi_1(t) - \phi_2(t)\| \le l_1(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } t \in [\alpha,\theta_k]\setminus\bigcup_{i=1}^{k-1}\left(\zeta_i^{(1)}, \zeta_i^{(2)}\right] \tag{31}$$

and

$$\zeta_i^{(2)} - \zeta_i^{(1)} \le \mu l_2(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } i = 1,2,\ldots,k-1, \tag{32}$$

where $l_1(\mu)$ and $l_1(\mu)$ are some bounded functions, and without any loss of generality $\zeta_i^{(2)} \ge \zeta_i^{(1)}$ for $i = 1,2,\ldots,p$.

Now we continue $\phi_j(t)$ as a solution of the following equation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = A(t)\phi(t) + \int_\alpha^t K(t,s)\phi(s)\,\mathrm{d}s + C(t)u_j(t) + f(t) + \mu g(t,\phi,u_j,\mu)$$

with initial condition $\phi(\theta_k) = \eta_j(\theta_k)$ until it meets a surface $t = \theta_k + \mu\tau_k(x,\mu)$ when $t = \zeta_k^{(j)}$. It is clear that for $t \in \left[\theta_k, \zeta_k^{(j)}\right]$,

$$\phi_j(t) = \eta_j(\theta_k) + \int_{\theta_k}^t \Bigg[A(s)\phi_j(s) + \int_\alpha^s K(t,\sigma)\phi_j(\sigma)\,\mathrm{d}\sigma + C(t)u_j(t) + f(t)$$
$$+\mu g(t,\phi_j(s),u_j(s),\mu)\Bigg]\,\mathrm{d}s. \tag{33}$$

Assume that $|\phi_j(t)| < H$ for all $t \in [\alpha,\theta_k]$. We claim that this inequality is also valid for all $t \in \left[\theta_k, \zeta_k^{(j)}\right]$. If it is not true for the first time at some point $t^* \in \left(\theta_k, \zeta_k^{(j)}\right]$, then using (33) we have $|\phi_j(t)| \le h + \mu_3 m_3 \gamma(\mu_3) < H$ for all $t \in [\theta_k, t^*]$, and in particular $|\phi_j(t^*)| < H$. This contradiction verifies our claim.

Now using (31)–(33) we obtain

$$\|\phi_1(t) - \phi_2(t)| \le l_3(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } t \in \left[\theta_k, \zeta_k^{(1)}\right], \tag{34}$$

where $l_3(\mu) = (1 + l_1(\mu)(\beta - \alpha) + p l_2(\mu)\gamma(\mu) + m_1 + \mu L)(1 - \mu m_3(m_1(1+\mu m_3) + \mu L))^{-1}$.

On the other hand, since

$$\zeta_k^{(2)} - \zeta_k^{(1)} = \mu\tau_k(\phi_2(\zeta_k^{(2)}),\mu) - \mu\tau_k(\phi_1(\zeta_k^{(1)}),\mu)$$
$$\le \mu L|\phi_2(\zeta_k^{(2)}) - \phi_2(\zeta_k^{(1)}) - \phi_2(\zeta_k^{(1)}) + \phi_1(\zeta_k^{(1)})|,$$

we have

$$\zeta_k^{(2)} - \zeta_k^{(1)} \le \mu l_4(\mu)\|\varphi_1 - \varphi_2\|, \tag{35}$$

where $l_4(\mu) = \mu L l_3(\mu)(1 - \mu L\gamma(\mu))^{-1}$.

Now we may consider S_k given by (24) and write

$$S_k(\eta_1, u_1, v_k^{(1)}, \mu) - S_k(\eta_2, u_2, v_k^{(2)}, \mu)$$
$$= B_k(\phi_1(\zeta_k^{(1)}) - \eta_1(\theta_k) - \phi_2(\zeta_k^{(1)}) + \eta_2(\theta_k)) + B_k(\phi_2(\zeta_k^{(1)}) - \phi_2(\zeta_k^{(2)}))$$
$$+ \sum_{\alpha < \zeta_j \le \zeta_k} [D_{kj}(\phi_1(\zeta_j^{(1)}) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(1)}) + \eta_2(\theta_j)]$$
$$+ D_{kj}(\phi_2(\zeta_j^{(1)}) - \phi_2(\zeta_j^{(2)}))]$$
$$+ \mu(W_k(\phi_1(\zeta_k^{(1)}, v_k^{(1)}, \mu) - W_k(\phi_2(\zeta_k^{(2)}, v_k^{(2)}, \mu)). \tag{36}$$

In view of (33),

$$\phi_1(\zeta_j^{(1)}) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(2)}) + \eta_2(\theta_j)$$
$$= \int_{\theta_k}^{\zeta_k^{(1)}} \Bigg\{A(s)[\phi_1(s) - \phi_2(s)] + \int_\alpha^s K(s,\sigma)[\phi_1(\sigma) - \phi_2(\sigma)]\,\mathrm{d}\sigma$$

$$+ C(s)\,[u_1(s) - u_2(s)] + \mu\,[g(s,\phi_1(s),u_1(s),\mu) - g(s,\phi_2(s),u_2(s),\mu)]\}\,\mathrm{d}s$$

$$+ \int_{\zeta_k^{(1)}}^{\zeta_k^{(2)}} \left\{ A(t)\phi_2(t) + \int_\alpha^t K(t,s)\phi_2(s)\,\mathrm{d}s + C(t)u_2(t) + f(t) \right.$$

$$\left. + \mu g(t,\phi_2(t),u_2(t),\mu) \right\} \mathrm{d}t$$

and therefore by using (34) and (35), we have

$$|\phi_1(\zeta_j^{(1)}) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(2)}) + \eta_2(\theta_j)| \le \mu l_5(\mu)\|\varphi_1 - \varphi_2\|, \tag{37}$$

where $l_5(\mu) = m_3(m_1(l_3(\mu) + l_1(\mu)(\beta - \alpha) + 1) + pl_3(\mu)\gamma(\mu) + \mu L) + l_4(\mu)\gamma(\mu)$.

It follows from (34)–(37) that

$$|S_k(\eta_1, u_1, v_k^{(1)}, \mu) - S_k(\eta_2, u_2, v_k^{(2)}, \mu)| \le \mu l^{(1)}(\mu)\|\varphi_1 - \varphi_2\|,$$

where $l^{(1)}(\mu) = m_1(p+1)(l_5(\mu) + l_3(\mu)\mu l_4(\mu)\gamma(\mu))$ is a bounded function. Since there are only finitely many S_j, it is clear that for some constant $L_1(\mu)$

$$|S_i(\eta_1, u_1, v_i^{(1)}, \mu) - S_i(\eta_2, u_2, v_i^{(2)}, \mu)| \le \mu L^{(1)}(\mu)\|\varphi_1 - \varphi_2\| \tag{38}$$

for $i = 1, 2, \ldots, p$.

Similarly by using the representation of F_i given in (25) one can show that there is a bounded function $L^{(2)}(\mu)$ such that

$$\left| \sum_{\alpha < \theta_i < t} F_i(t, \eta_2, u_2, \mu) - \sum_{\alpha < \theta_i < t} F_i(t, \eta_1, u_1, \mu) \right| \le \mu L^{(2)}(\mu)\|\varphi_1 - \varphi_2\| \tag{39}$$

uniformly for all $t \in [\alpha, \beta]$.

Let $m_4 = \max_{t,s} |\Lambda(t,s)|$. Using (38) and (39) it follows that

$$|\psi(t,\varphi_1,\mu) - \psi(t,\varphi_2,\mu)| \le pm_4 L^{(1)}(\mu)\|\varphi_1 - \varphi_2\|$$

and

$$|\kappa(t,\varphi_1,\mu) - \kappa(t,\varphi_2,\mu)| \le (\beta - \alpha)m_4 L^{(2)}(\mu)\|\varphi_1 - \varphi_2\|.$$

Thus we have

$$|\kappa(t,\varphi_1,\mu) - \kappa(t,\varphi_2,\mu)| + |\psi(t,\varphi_1,\mu) - \psi(t,\varphi_2,\mu)| \le \mu L(\mu)\|\varphi_1 - \varphi_2\|,$$

where $L(\mu) = (\beta - \alpha)L^{(2)}(\mu) + pm_4 L^{(1)}(\mu)$.

Finally, letting

$$m_5 = \max\left\{ \max_t \|\Psi(t)\Psi^{-1}(\beta)\|, \max_t \|E^{\mathrm{T}}(t)\Psi^{-1}(\beta)\|, \max_i \|P_i^{\mathrm{T}}\Psi^{-1}(\beta)\| \right\}$$

one can verify that

$$\|\mathscr{P}(\varphi_1,\mu) - \mathscr{P}(\varphi_2,\mu)\| \le 2L(\mu)(1 + 6m_5)\|\varphi_1 - \varphi_2\|.$$

Thus, if we choose $\mu_0 \le \mu_4$ sufficiently small so that $2\mu L(\mu)(1 + 6m_5) < 1$ for all $\mu < \mu_0$, then the operator $\mu\mathscr{P}$ becomes contraction. So there is a fixed point φ^0 of the operator $\phi_0 + \mu\mathscr{P}(\varphi,\mu)$. It is easy to verify that φ^0 solves problem γ_μ, and therefore the proof is complete. □

Theorem 6. *Let* (c_1)–(c_5) *be satisfied. If* $\Psi(\beta)$ *be nonsingular, then* Σ_μ *is solvable.*

Proof. By Lemma 5, we can make sure that there is a control $\{\hat{u}, \hat{v}\}$ for which system (21) admits a solution $y(t)$ such that $y(\alpha) = a$ and $y(\beta) = b$.

Since (1) and (21) have the property Ω, system (1) with control $\{\hat{u}, \hat{v}\}$ has a solution $x(t)$ such that $x(t) = y(t)$ for all t except for $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$. Because α and β are not in $[\theta_i, \zeta_i]$ for any $i = 1, 2, \ldots, p$, the solution $x(t)$ also satisfies the boundary condition (2). This completes the proof. □

References

[1] H. Akça, L. Berezansky, E. Braverman, On linear integro-differential equations with integral impulsive conditions, Z. Anal. Anwendungen 15 (1996) 709–727.
[2] M.U. Akhmetov, N.A. Perestyuk, On comparison method for differential systems with pulse effect, Differential Equations 9 (26) (1990) 1079–1086.
[3] M.U. Akhmetov, A. Zafer, The controllability of boundary-value problems for quasilinear impulsive systems, Nonlinear Analysis 34 (1998) 1055–1065.
[4] L. Anokhin, L. Berezansky, E. Braverman, Stability of linear delay impulsive differential equations, Dyn. Systems Appl. 4 (1995) 173–188.
[5] S.A. Aysagaliev, K.O. Onaybaer, T.G. Mazakov, The controllability of nonlinear systems, Izv. Akad. Nauk. Kaz. SSR. Ser. Fiz. Mat 1 (1985) 83–87.
[6] S. Barnett, Introduction to Mathematical Control Theory, Clarendon Press, Oxford, 1975.
[7] A. Halanay, D. Wexler, Qualitative Theory of Impulse Systems, Editura Academiei Republici Socialiste Romania, Bucuresti, 1968.
[8] N.N. Krasovsky, Theory of Motion Control. Linear Systems, Nauka, Moscow, 1973.
[9] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[10] E.B. Lee, L. Markus, Foundations of Optimal Control Theory, Wiley, New York, 1967.
[11] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.