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The Control of the Boundary Value Problem for Linear Impulsive Integro-Differential Systems

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1. INTRODUCTION

Recently, there has been a lot of activity in the investigation of differential equations with solutions which undertake discontinuities or jumps at some definite instants [1–4]. At the same time, in the theory of impulsive differential equations there is a number of problems which are connected with the results of the theory of integral and integro-differential equations and which are not well investigated yet [3, 5–8]. The purpose of this paper is to consider a problem of finding conditions for controllability of the boundary value problem for a linear impulsive system of integro-differential equations. The problem of the control of the linear impulse system was considered in [9]. Moreover, the existence results and the integral representation of solutions of the second kind Volterra integro-summary equation and integro-differential equations are obtained with impulse actions at fixed moments of time. Necessary and sufficient conditions of the solvability of the boundary value problem for linear impulsive systems of integro-differential equations are defined.

2. PRELIMINARIES

Let α and β be fixed real numbers such that $\alpha < \beta$ and p and r fixed positive integers. Denote the set of square integrable functions φ: $[\alpha, \beta] \to R^r$ by $\mathbf{L}_2^r[\alpha, \beta]$ and the set of all finite sequences $\{\xi_i\}$, $\xi_i \in R^r$, $i = \overline{1, p}$,

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by $\mathbf{D}^r[1, p]$. We define a space $\mathbf{\Pi}^r[\alpha, \beta] = \mathbf{L}_2^r[\alpha, \beta] \times \mathbf{D}^r[1, p]$ and denote its elements as $\{\varphi, \xi\}$. Let

$$\langle \{\varphi, \xi\}, \{\omega, \nu\} \rangle = \int_\alpha^\beta (\varphi, \omega)\, dt + \sum_{i=1}^{p} (\xi_i, \nu_i)$$

be an inner product in $\mathbf{\Pi}^r[\alpha, \beta]$, where $(\ ,\ )$ is the euclidean scalar product in R^r.

Let (H_0) θ_i, $i = \overline{1, p}$, strictly increasing the interval (α, β) sequence of real numbers.

We denote by $\mathbf{PAC}[\alpha, \beta]$ the space of piecewise absolutely continuous, continuous on the left functions $x\colon [\alpha, \beta] \to R^n$ with jumps at the points θ_i, $i = \overline{1, p}$. The two following assertions are analogous to Fubini's theorem [8].

LEMMA 1. *Let D_{ij}, $i, j = \overline{1, p}$ be constant matrices of size $n \times n$, $\{\xi_i\} \in \mathbf{D}^n[1, p]$ and the assumption (H_0) hold. Then the following equality is valid for any t, $\alpha < t < \beta$:*

$$\sum_{\alpha < \theta_i < t}\ \sum_{\alpha < \theta_j \le \theta_i} D_{ij} \xi_j = \sum_{\alpha < \theta_i < t} \left[\sum_{\theta_i \le \theta_j < t} D_{ji} \right] \xi_i. \tag{1}$$

This statement is proved by a regrouping of addends.

LEMMA 2. *Suppose that the matrix $K(t, s)$ of size $n \times n$ is square integrable on the interval $\alpha < s < \beta$, $\varphi_i(t) \in \mathbf{L}_2^n[\alpha, \beta]$, $i = \overline{1, n}$, and the assumption (H_0) holds.*

The following equality is valid for any t, $\alpha < t < \beta$:

$$\int_\alpha^t K(t, s) \sum_{\alpha < \theta_i < s} \varphi_i(s)\, ds = \sum_{\alpha < \theta_i < t} \left[\int_{\theta_i}^t K(t, s) \varphi_i(s)\, ds \right]. \tag{2}$$

Proof. Construct the functions $\Phi_i(t)$, $i = \overline{1, p}$, which are equal to 0 if $t \le \theta_i$ and they are equal to $\varphi_i(t)$ if $t > 0$. Then the left hand side of (2) is equal to

$$\begin{aligned}\int_\alpha^t K(t, s) \left[\sum_{\alpha < \theta_i < t} \Phi_i(s) \right] ds &= \sum_{\alpha < \theta_i < t} \int_\alpha^t [K(t, s) \Phi_i(s)\, ds] \\ &= \sum_{\alpha < \theta_i < t} \left[\int_{\theta_i}^t K(t, s) \varphi_i(s)\, ds \right].\end{aligned}$$

The proof of the lemma is complete.

We now consider the integral equation having the form

$$x(t) = \int_{\alpha}^{t} G(t,s)x(s)\,ds + \sum_{\alpha<\theta_i<t} M_i(t)x(\theta_i) + \sum_{\alpha<\theta_i<t} N_i(t)x(\theta_i+) + f(t) + \sum_{\alpha<\theta_i<t} I_i, \tag{3}$$

under the following assumptions:

(H_1) M_i and N_i are function matrices of size $n \times n$, $f(t)$: $[\alpha, \beta] \to R^n$;

(H_2) the matrix $G(t,s)$: $[\alpha, \beta] \times [\alpha, \beta] \to R^n$ is square integrable;

(H_3) the lines of matrices $M_i(t)$, $N_i(t)$ and the function $f(t)$: $[\alpha, \beta] \to R^n$ belong to $\mathbf{PAC}[\alpha, \beta]$;

(H_4) for all $i = \overline{1, p}$ the inequality $\det(I - N_i(\theta_i)) \neq 0$ is valid.

THEOREM 1. *Let the conditions* (H_0)–(H_4) *hold. The system* (3) *has the unique solution* $x(t) \in \mathbf{PAC}[\alpha, \beta]$. *The solution* $x(t)$ *has the following representation*:

$$x(t) = \int_{\alpha}^{t} P_1(t,s)f(s)\,ds + \sum_{\alpha<\theta_i<t} Q_i(t)I_i + \sum_{\alpha<\theta_i<t} P_2^i(t,s)f(\theta_i) + f(t) + \sum_{\alpha<\theta_i<t} I_i, \tag{4}$$

where $Q_i(t)$, $P_2^i(t,s)$, $i = \overline{1, p}$, *and* $P_1(t,s)$ *are piecewise continuous function matrices of size* $n \times n$.

Proof. Let $R(t,s)$ be the resolvent of the Volterra integral equation whose kernel is $G(t,s)$. Then, using Lemmas 1 and 2 we can obtain that (3) is equivalent to the system

$$\begin{aligned} x(t) = {} & \sum_{\alpha<\theta_i<t} \left[\int_{\theta_i}^{t} R(t,s)M_i(s)\,ds + M_i(t) \right] x(\theta_i) \\ & + \sum_{\alpha<\theta_i<t} \left[\int_{\theta_i}^{t} R(t,s)N_i(s)\,ds + N_i(t) \right] x(\theta_i+) \\ & + \sum_{\alpha<\theta_i<t} \left[\int_{\theta_i}^{t} R(t,s)\,ds \right] I_i \\ & + \int_{\theta_i}^{t} R(t,s)f(s)\,ds + f(t) + \sum_{\alpha<\theta_i<t} I_i. \end{aligned} \tag{5}$$

Let

$$M_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j, s) M_i(s)\, ds + M_i(\theta_j),\ N_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j, s) N_i(s)\, ds + N_i(\theta_j),$$

$$\pi_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j, s)\, ds + I,$$

where I is the identity $n \times n$ matrix. Then from (5) it follows that

$$x(\theta_j) = \sum_{\alpha < \theta_i < \theta_j} \left[M_{ij} x(\theta_i) + N_{ij} x(\theta_i +) \right] + \sum_{\alpha < \theta_i < \theta_j} \pi_{ij} I_i$$
$$+ \int_{\alpha}^{\theta_j} R(\theta_j, s) f(s)\, ds + f(\theta_j). \tag{6}$$

Using (5) and (6) we obtain that

$$x(\theta_j +) = \left(I - N_j(\theta_j)\right)^{-1} \left(I - M_j(\theta_j)\right) x(\theta_j) + \left(I - N_j(\theta_j)\right)^{-1} I_j. \tag{7}$$

Expressions (6) and (7) imply that $x(\theta_j)$ and $x(\theta_j +)$ are defined as solutions of a linear nonhomogeneous algebraic system. A nonhomogeneous part of this system is a linear combinations of vectors

$$\int_{\alpha}^{\theta_i} R(t, s) f(s)\, ds,\ f(\theta_i),\ I_i,\ i = \overline{1, p}. \tag{8}$$

Consequently, the vectors $x(\theta_j)$и$x(\theta_j +)$ are also linear combinations of vectors (8) with matrix coefficients.

Substituting their values in (5) we obtain that the solution of (3) has a form (4). The proof of the theorem is complete.

3. THE BOUNDARY VALUE PROBLEMS

Consider the impulse system of integro-differential equations

$$\begin{aligned} dx/dt &= A(t)x + f(t) + \int_{\alpha}^{t} K(t, s) x(s)\, ds,\ t \neq \theta_i, \\ \Delta x(\theta_i) &= B_i x(\theta_i) + \sum_{\alpha < \theta_j \le \theta_i} D_{ij} x(\theta_j) + I_i, \end{aligned} \tag{9}$$

where $\Delta x(\theta_j) \equiv x(\theta_j +) - x(\theta_j)$, under the following assumptions:

(C_1) A and K are function matrices of size $n \times n$;

(C_2) the matrix $K(t, s)$: $[\alpha, \beta] \times [\alpha, \beta] \to R^n \times R^n$ is square integrable;

(C_3) the lines of the matrix $A(t)$ and the function $f(t)\colon [\alpha, \beta] \to R^n$ belong to $\mathbf{L}_2^n[\alpha, \beta]$;

(C_4) D_{ij}, B_i, $i, j = \overline{1, p}$ are constant matrices of size $n \times n$;

(C_5) $\{f, I\} \in \mathbf{\Pi}^n[\alpha, \beta]$.

We consider (9) with the boundary condition

$$x(\alpha) = a,\ x(\beta) = b,\ a, b \in R^n. \tag{10}$$

THEOREM 2. *Let conditions* (H_0) *and* (C_1)–(C_5) *hold. Then for any* $x_0 \in R^n$ *there exists a unique solution* $x(t) \in \mathbf{PAC}[\alpha, \beta]$ *of* (9) *and is defined on the interval* $[\alpha, \beta]$ *such that* $x(\alpha) = x_0$.

Proof. Applying differentiating and checking the conditions of discontinuities one can verify that the following integro-differential equation

$$x(t) = \int_\alpha^t A(s)x(s)\,ds + \int_\alpha^t \left[\int_\alpha^\sigma K(\sigma, s)x(s)\,ds\right] d\sigma + \sum_{\alpha < \theta_i < t} B_i x(\theta_i)$$

$$+ \int_\alpha^t f(s)\,ds + \sum_{\alpha < \theta_i < t}\ \sum_{\alpha < \theta_j \le \theta_i} D_{ij} x(\theta_i) + \sum_{\alpha < \theta_i < t} I_i \tag{11}$$

is equivalent to (9).

Let

$$\Psi(t, s) = A(s) + \int_s^t K(\sigma, s)\,d\sigma,\ \Phi_i = B_i + \sum_{\theta_i \le \theta_j < t} D_{ji},$$

$$F(t) = x_0 + \int_\alpha^t f(s)\,ds.$$

Applying Fubini's theorem and Lemmas 1 and 2 one can rewrite the last equation as

$$x(t) = \int_\alpha^t \Psi(t, s)x(s)\,ds + \sum_{\alpha < \theta_i < t} \Phi_i x(\theta_i) + \sum_{\alpha < \theta_i < t} I_i + F(t). \tag{12}$$

Equation (12) is a system like (3). Therefore, according to Theorem 1, (12) shows the unique solution $x(t) \in \mathbf{PAC}[\alpha, \beta]$. The proof of the theorem is complete.

We now consider the system of integro-differential equations

$$\partial h/ds = -h(t, s)A(s) - \int_s^t h(t, \sigma)K(\sigma, s)\,d\sigma,\ s \ne \theta_i,$$

$$\Delta h(t, \theta_i) = -h(t, \theta_i)B_i(I + B_i)^{-1} - \sum_{\theta_i \le \theta_j < t} h(t, \theta_j +)D_{ji}(I + B_i)^{-1}, \tag{13}$$

where $h \in R^n$, h is the row vector, $t \in [\alpha, \beta]$, $A(t)$, $K(t,s)$, D_{ji}, B_i are defined as in (9), $\Delta h(t, \theta_i) \equiv h(t, \theta_i +\ ) - h(t, \theta_i)$.

Let

(C_6) $\det(I - D_{jj}(I + B_j)^{-1}) \neq 0$ for all $j = \overline{1, p}$.

THEOREM 3. *Let conditions* (H_0), (C_1)–(C_4), *and* (C_6) *hold. Then* (13) *has the unique solution* $h(t,s)$ *such that* $h(t,t) = h_0$, $h_0 \in R^n$.

Proof. System (16) is equivalent to the integro-differential equation

$$h(t,s) = h(t,t) + \int_s^t \left[h(t,\sigma) A(\sigma) + \int_\sigma^t h(t,v) K(v,\sigma)\, dv \right] d\sigma + \sum_{s \leq \theta_i < t} \left[-h(t,\theta_i) B_i (I + B_i)^{-1} + \sum_{\theta_i \leq \theta_j < t} h(t, \theta_j +) D_{ji} (I + B_i)^{-1} \right]. \tag{14}$$

Using the theorem of Fubini and Lemmas 1 and 2 we obtain that (14) is equivalent to the system

$$h(t,s) = h_0 + \int_s^t h(t,\sigma) \left[A(\sigma) + \int_s^t K(\sigma, v)\, dv \right] d\sigma + \sum_{s < \theta_i < t} h(t,\theta_i) B_i (I + B_i)^{-1} + \sum_{s \leq \theta_i < t} h(t, \theta_i +) \left[\sum_{s \leq \theta_j \leq \theta_i} D_{ij} (I + B_i)^{-1} \right]. \tag{15}$$

Comparing (15) and (3) it is not difficult to see that for (15) we can use all arguments of Theorem 1. Therefore, (13) has the unique solution $h(t,s)$, $h(t,t) = h_0$. The proof is complete.

Let $H(t,s) = \operatorname{colon}(H_1, H_2, H_3, \dots, H_n)$, $H(t,t) = I$, H_j, $j = \overline{1, n}$, are solutions of (14).

THEOREM 14. *Let* $x(t) = x(t, \alpha, x_0)$ *be a solution of the Cauchy problem for* (9). *Then* $x(t)$ *has the representation*

$$x(t) = H(t,\alpha) x_0 + \int_\alpha^t H(t,s) f(s)\, ds + \sum_{\alpha < \theta_j < t} H(t, \theta_j +) I_j. \tag{16}$$

Proof. Denote $\varphi(s) = H(t, s)x(s)$, where $x(t) = x(t, \alpha, x_0)$ is the solution of (9). It is known [1] that

$$\varphi(t) - \varphi(\alpha) = \int_\alpha^t \varphi'(s)\,ds + \sum_{\alpha < \theta_i < t} \Delta\varphi(\theta_i). \tag{17}$$

Consider each addends in (17).

1. Since

$$\begin{aligned}\Delta\varphi(\theta_j) &= H(t, \theta_j +)x(\theta_j +) - H(t, \theta_j)x(\theta_j)\\ &= H(t, \theta_j)\Delta x(\theta_j) + \Delta H(t, \theta_j)x(\theta_j +).\end{aligned} \tag{18}$$

Adding both parts of (18) with respect to i which satisfies $\alpha < \theta_i < t$ we obtain that

$$\begin{aligned}&\sum_{\alpha < \theta_i < t} \Delta\varphi(\theta_i)\\ &\quad = \sum_{\alpha < \theta_i < t} \left[\Delta H(t, \theta_i)(I + B_i) + H(t, \theta_i)B_i\right]x(\theta_i)\\ &\qquad + \sum_{\alpha < \theta_i < t}\ \sum_{\alpha < \theta_j \le \theta_i} H(t, \theta_i +)D_{ij}x(\theta_j) + \sum_{\alpha < \theta_i < t} H(t, \theta_i +)I_i\\ &\quad = \sum_{\alpha < \theta_i < t} \Bigg[\Delta H(t, \theta_i)(I + B_i) + H(t, \theta_i)B_i\\ &\qquad\qquad + \sum_{\theta_i \le \theta_i < t} H(t, \theta_j +)D_{ji}\Bigg]x(\theta_i)\\ &\qquad + \sum_{\alpha < \theta_i < t} H(t, \theta_i +)I_i.\end{aligned}$$

Since $H(t, s)$ is the solution of (13), from the last expression it follows that

$$\sum_{\alpha < \theta_i < t} \Delta\varphi(\theta_i) = \sum_{\alpha < \theta_i < t} H(t, \theta_i +)I_i. \tag{19}$$

2. Differentiating the expression $\varphi(s) = H(t, s)x(s)$ we obtain that

$$\begin{aligned}\varphi'(s) &= \partial H/\partial s\, x(s) + H(t, s)\\ &\qquad \times\left[A(s)x(s) + f(s) + \int_\alpha^s K(s, v)x(v)\,dv\right].\end{aligned}$$

By Fubini's theorem we find that

$$\int_{\alpha}^{t}\varphi'(s)\,ds = \int_{\alpha}^{t}[\partial H/\partial s + H(t,s)A(s)]x(s)\,ds$$
$$+\int_{\alpha}^{t}H(t,s)\left[\int_{\alpha}^{s}K(s,v)x(v)\,dv\right]ds + \int_{\alpha}^{t}H(t,s)f(s)\,ds$$
$$= \int_{\alpha}^{t}\left[\partial H/\partial s + H(t,s)A(s) + \int_{\alpha}^{t}H(t,v)K(v,s)\,dv\right]$$
$$\times x(s)\,ds + \int_{\alpha}^{t}H(t,s)f(s)\,ds.$$

From (13) and the last expression it follows that

$$\int_{\alpha}^{t}\varphi'(s)\,ds = \int_{\alpha}^{t}H(t,v)f(v)\,dv. \tag{20}$$

In view of (17), by adding (19) and (20) we find the following representation:

$$\varphi(t) - \varphi(\alpha) = \int_{\alpha}^{t}H(t,s)f(s)\,ds + \sum_{\alpha<\theta_j<t}H(t,\theta_j+)I_j.$$

Since $H(t,t) = I$ the last equalities implies (16). The proof is complete.

Consider the system (9) and the boundary conditions

$$x(\alpha) = 0,\ x(\beta) = 0. \tag{21}$$

THEOREM 5. *The boundary value problem* (9), (21) *is solved if and only if for all* $j = \overline{1,p}$, *the condition*

$$\left\langle\left\{H_j^T(\beta,s), H_j^T(\beta,\theta_j+)\right\},\ \left\{f(s), I_j\right\}\right\rangle = 0 \tag{22}$$

is valid.

Theorem 5 *is a simple corollary of Theorem* 4.

THEOREM 6. *The boundary value problem* (9), (10) *is solved if and only if for all* $j = \overline{1,p}$ *the condition*

$$\left\langle\left\{H_j^T(\beta,s), H_j^T(\beta,\theta_j+)\right\},\left\{f(s), I_j\right\}\right\rangle = H_j(\beta,\beta)b - H_j(\beta,\alpha)a \tag{23}$$

is true.

Proof. Let $\varphi(t)$ be Lagrange's polynomial such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$, $\varphi(\theta_j) = 0$, $j = \overline{1, p}$. Replacing $x(t)$ by $y(t) + \varphi(t)$ in (9), we see that $y(t)$ satisfies

$$\begin{aligned} dy/dt &= A(t)y + f(t) + \int_\alpha^t K(t,s)y(s)\,ds \\ &\quad - \left[\varphi'(t) - A(t)\varphi(t) - \int_\alpha^t K(t,s)\varphi(s)\,ds\right],\ t \neq \theta_i, \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{\alpha < \theta_j \leq \theta_i} D_{ij} y(\theta_j) + I_i, \end{aligned} \tag{24}$$

with the boundary condition

$$y(\alpha) = 0,\ y(\beta) = 0. \tag{25}$$

According to Theorem 5 the problem (24), (25) is solved if and only if

$$\left\langle \left\{ H_j^T(\beta, s), H_j^T(\beta, \theta_j +) \right\}, \left\{ F(s), I_j \right\} \right\rangle = 0,\ j = \overline{1, p}, \tag{26}$$

where

$$F(t) = -\varphi'(t) + A(t)\varphi(t) + \int_\alpha^t K(t,s)\varphi(s)\,ds.$$

Integrating by part in (26) we obtain the complete proof of the theorem.

4. THE CONTROLLABILITY OF THE BOUNDARY VALUE PROBLEM

Consider the nonhomogeneous boundary value problem

$$\begin{aligned} dx/dt &= A(t)x + \int_\alpha^t K(t,s)x(s)\,ds + C(t)u(t) + f(t),\ t \neq \theta_i, \\ \Delta x(\theta_i) &= B_i x(\theta_i) + \sum_{\alpha < \theta_j \leq \theta_i} D_{ij} x(\theta_j) + Q_i v_i + I_i, \end{aligned} \tag{27}$$

$$x(\alpha) = a,\ x(\beta) = b,\ a, b \in R^n. \tag{28}$$

In (27), (28) $x \in R^n$, the matrices A, K, B_i, $i = \overline{1, p}$, the same in (9), the matrices $C(t)$ and Q_i, $i = \overline{1, p}$, of size $n \times m$, m is the fixed natural number, the columns of $C(t)$ are functions of $\mathbf{L}_2^n[\alpha, \beta]$, Q_i, $i = \overline{1, p}$, are constant matrices, and solutions of (27) are elements of $\mathbf{PAC}[\alpha, \beta]$.

If for each $\{f, I\} \in \mathbf{\Pi}^n[\alpha, \beta]$ and every $a, b \in R^n$ there exists a control $\{u, v\} \in \mathbf{\Pi}^m[\alpha, \beta]$ such that the problem (27) and (28), which we shall refer to as γ_1, has a solution, then we will say that the control problem γ_1 is solvable.

We also consider the problem γ_1 in the case $a = 0$, $b = 0$. This problem is to be called the control problem γ_2.

LEMMA 3. *γ_1 is solvable if and only if γ_2 is.*

Proof. Let γ_1 be solvable. Since γ_2 is a particular case of γ_1, γ_2 is solvable. Suppose that γ_2 is solvable. Let $\varphi(t)$ be the Lagrange polynomial such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Replacing $x(t)$ by $y(t) + \varphi(t)$ in (27), (28) we see that $y(t)$ satisfies

$$\begin{aligned} dy/dt &= A(t)x + \int_\alpha^t K(t,s)y(s)\,ds + C(t)u(t) \\ &\quad + [f(t) - \varphi'(t) + A(t)\varphi(t)], \quad t \neq \theta_i, \qquad (29) \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{\alpha < \theta_j \leq \theta_i} D_{ij} y(\theta_j) + Q_i v_i + \left[I_i + B_i \varphi(\theta_j)\right], \\ y(\alpha) &= 0, \quad y(\beta) = 0. \end{aligned}$$

This problem is solvable due to our assumption. Thus, the lemma is proved.

THEOREM 7. *γ_1 is solvable if and only if the trivial solution of* (13) *is the unique solution of*

$$\left\langle \{Cu, Qv\}, \{h^T, h^T\} \right\rangle = 0, \ \forall \{u, v\} \in \mathbf{\Pi}^m[\alpha, \beta]. \qquad (30)$$

Proof. *Sufficiency*. Let $h(t,t) = h, h \in R^n$ be a solution of (13), then $h(t,s) = hH(t,s)$. Therefore, according to conditions of the theorem, the infinite system

$$\left\langle \{Cu, Qv\}, \left\{H^T(\beta, s)h^T, H^T(\beta, \theta_i +)h^T\right\} \right\rangle = 0, \ \forall \{u, v\} \in \mathbf{\Pi}^m[\alpha, \beta] \qquad (31)$$

has only the trivial solution $h = 0$.

We shall show that there are n elements $\{u^\kappa, v^\kappa\} \in \mathbf{\Pi}^m[\alpha, \beta]$, $\kappa = \overline{1, n}$, such that the matrix

$$N = \left\langle \{Cu^\kappa, Qv^\kappa\}, \left\{H_j^T, H_j^T\right\} \right\rangle_{jk}, \ j, \kappa = \overline{1, n} \qquad (32)$$

is nonsingular.

Suppose the contrary. Without any loss of generality we may assume that the last row of matrix N can be written as a linear combination of other rows. Denote by h^0 a nontrivial solution of the system

$$\left\langle \{Cu^\kappa, Qv^\kappa\}, \{H^T h^T, H^T h^T\}\right\rangle = 0, \ \kappa = \overline{1, n}. \tag{33}$$

It is true that for every $\{u, v\}$ there are numbers μ_k, $k = \overline{1, n-1}$ such that

$$\left\langle \{Cu, Dv\}, \left\{H_j^T, H_j^T\right\}\right\rangle = \sum_{k=1}^{n-1} \mu_k \left\langle \{Cu^k, Dv^k\}, \left\{H_j^T, H_j^T\right\}\right\rangle, \ j = \overline{1, n}.$$

Therefore (33) implies (31) with $h = h^0$.
Really. Let $h^0 = (h_1^0, h_2^0, h_3^0, \dots, h_n^0)$. Then it is possible to write

$$h^0 H = \sum_{j=1}^{n} h_j^0 H_j(t, s). \tag{34}$$

Hence,

$$\begin{aligned}
&\left\langle \{Cu, Qv\}, \{H^T h^{0T}, H^T h^{0T}\}\right\rangle \\
&\quad = \left\langle \{Cu, Qv\}, \left\{ \sum_{j=1}^{n} H_j^T(t, s) h_j^{0T}, \ \sum_{j=1}^{n} H_j^T(t, s) h_j^{0T} \right\}\right\rangle \\
&\quad = \sum_{j=1}^{n} h_j^0 \left\langle \{Cu, Qv\}, \left\{H_j^T, H_j^T\right\}\right\rangle \\
&\quad = \sum_{j=1}^{n} h_j^0 \sum_{k=1}^{n-1} \mu_k \left\langle \{Cu^k, Qv^k\}, \left\{H_j^T, H_j^T\right\}\right\rangle \\
&\quad = \sum_{k=1}^{n-1} \mu_k \sum_{j=1}^{n} h_j^0 \left\langle \{Cu^k, Qv^k\}, \left\{H_j^T, H_j^T\right\}\right\rangle \\
&\quad = \sum_{k=1}^{n-1} \mu_k \left\langle \{Cu^k, Qv^k\}, \{H^T h^{0T}, H^T h^{0T}\}\right\rangle = 0.
\end{aligned}$$

Thus, $h^0 H$ is a nontrivial solution of (13), which satisfies (31). Consequently, the matrix N must be nonsingular.

We now consider the following boundary value problem:

$$dx/dt = A(t)x + \int_\alpha^t K(t,s)x(s)\,ds - C(t)\sum_{k=1}^n m_k u^k + f(t),\ t \neq \theta_i, \tag{35}$$

$$\Delta x(\theta_i) = B_i x(\theta_i) + \sum_{\alpha < \theta_j \le \theta_i} D_{ij} x(\theta_j) - Q_i \sum_{k=1}^n m_k v_i^k + I_i,$$

$$x(\alpha) = 0,\ x(\beta) = 0, \tag{36}$$

where $\{u^\kappa, v^\kappa\} \in \mathbf{\Pi}^m[\alpha, \beta]$, $\kappa = \overline{1,n}$ were defined as above. Since the matrix N is nonsingular, the system

$$\sum_{k=1}^n \left\langle \{Cu^k, Qv^k\}, \left\{H_j^T, H_j^T\right\}\right\rangle m_\kappa = \left\langle \{f, I\}, \left\{H_j^T, H_j^T\right\}\right\rangle,\ j = \overline{1,n}$$

has a solution and therefore according to Theorem 2, (35), (36) is a solvable problem.

Necessity. Suppose on the contrary that γ_1 is a solvable problem and there is a nontrivial solution h of (13) which satisfies (30). It is easy to show that there exists $\{f, I\} \in \mathbf{\Pi}^n[\alpha, \beta]$, satisfying $\langle \{f, I\}, \{h^T, h^T\}\rangle \neq 0$. We fix this element. By adding the last inequality and expression (30) we obtain the pair $\{f, I\} \in \mathbf{\Pi}^n[\alpha, \beta]$, which satisfies the relation $\langle \{Cu + f, Qv + I\}, \{h^T, h^T\}\rangle \neq 0$ for all $\{u, v\}$. Since this contradicts Theorem 2 the proof is complete.

The last theorem implies that problems γ_1 and γ_2 are solvable if and only if the system

$$\left\langle \{Cu, Qv\}, \left\{H(\beta, s)^T h^T, H(\beta, \theta_i +)^T h^T\right\}\right\rangle = 0,\ \forall \{u, v\} \in \mathbf{\Pi}^m[\alpha, \beta] \tag{37}$$

has only a trivial solution with respect to $h \in R^n$.

Moreover, problems γ_1 and γ_2 are solvable if and only if for any $t \in [\alpha, \beta]$, $i = \overline{1, p}$, the relations

$$\det(H(\beta, t)C(t)) \neq 0,\ \det(H(\beta, \theta_i)Q_i) \neq 0 \tag{38}$$

are true.

Let $\mathbf{\Gamma}$ be the Gram matrix of elements $\{H_jC, H_j(\beta, \theta_j+)Q_j\}$, $j = \overline{1, n}$, i.e.,

$$\mathbf{\Gamma} = \int_\alpha^\beta H(\beta, t)C(t)C^T(t)H^T(\beta, t)\,dt + \sum_{k=1}^{p} H(\beta, \theta_i+)Q_iQ_i^TH^T(\beta, \theta_i+).$$

THEOREM 8. *γ_1 is solvable if and only if Gram's matrix $\mathbf{\Gamma}$ is nonsingular.*

Proof. By Theorem 7 the system (30) has only the trivial solution $h = 0$. Setting $\{u, v\} = \{C^T(t)H^T(\beta, t)h^T, Q_i^TH^T(\beta, \theta_i)h^T\}$ in this equation we find that the system $h\mathbf{\Gamma} = 0$ has also only a trivial solution. If the equation $h\mathbf{\Gamma} = 0$ has only a trivial solution then system (30) has only the solution $h = 0$. The proof is complete.

THEOREM 9. *The control $\{u, v\} \in \mathbf{\Pi}^m[\alpha, \beta]$ solves γ_1 if and only if the condition*

$$\int_\alpha^\beta H(\beta, t)[C(t)u(t) + f(t)]\,dt + \sum_{k=1}^{p} H(\beta, \theta_i+)[Q_iv_i + I_i] = H(\beta, \beta)b - H(\beta, \alpha)a \tag{39}$$

holds.

Let

$$\mathbf{K} = \mathbf{\Gamma}^{-1}\left\{H(\beta, \beta)b - H(\beta, \alpha)a - \int_\alpha^\beta H(\beta, t)f(t)\,dt - \sum_{k=1}^{p} H(\beta, \theta_i+)I_i\right\},$$

$$S(t) = H(\beta, t)C(t) \text{ and } P_i = H(\beta, \theta_i+)Q_i.$$

THEOREM 10. *Suppose that γ_1 is solvable. Then the control $\{U, V\}$, where $U = S^T(t)\mathbf{K}$, $V_i = P_i^T\mathbf{K}$, solves γ_1.*

Proof. Let $\varphi(t)$ be the Lagrange polynomial such that $\varphi(\theta_i) = 0$, $i = \overline{1, p}$. Replacing $x(t)$ by $z(t) + \varphi(t)$ in γ_1, we see that z satisfies

$$dz/dt = A(t)z + \int_\alpha^t K(t, s)z(s)\,ds + C(t)u(t) + f(t) + [\varphi'(t) - A(t)\varphi(t)],\ t \neq \theta_i, \tag{40}$$

$$\Delta z(\theta_i) = B_iz(\theta_i) + \sum_{\alpha < \theta_j \leq \theta_i} D_{ij}z(\theta_j) + Q_iv_i + I_i,$$

$$z(\alpha) = z(\beta) = 0. \tag{41}$$

According to Theorem 5 the problem (40), (41) is solvable if and only if

$$\int_\alpha^\beta H(\beta,t)[C(t)u(t)+f(t)]\,dt+\sum_{k=1}^p H(\beta,\theta_i)[Q_iv_i+I_i]$$
$$=\int_\alpha^\beta H(\beta,t)[\varphi'(t)-A(t)\varphi(t)]\,dt,\ \forall\{u,v\}\in\mathbf{\Pi}^m[\alpha,\beta].$$

Integration by part of the last expression we can rewrite it as

$$\int_\alpha^\beta H(\beta,t)[C(t)u(t)+f(t)]\,dt+\sum_{k=1}^p H(\beta,\theta_i)[Q_iv_i+I_i]$$
$$=H(\beta,\beta)b-H(\beta,\alpha)a. \tag{42}$$

Substituting expressions

$$U=C^T(t)H^T(\beta,t)h^T,\ W_i=Q_i^TH^T(\beta,\theta_i)h^T \tag{43}$$

in (42) we obtain the system of linear equations with respect to h. Using the solution of this system in (43) we obtain the expression for $\{U,V\}$. The proof is complete.

The control $\{U,V\}$ lets us describe the set of all controls solving γ_1.

THEOREM 11. *The control* $\{u,v\}$ *solves* γ_1 *if and only if it is of the form* $u=U+\xi$, $v_i=W_i+\nu_i$, *where* $\{\xi,\nu\}\in\mathbf{\Pi}^m[\alpha,\beta]$ *is orthogonal in* $\mathbf{\Pi}^m[\alpha,\beta]$ *to all columns of matrix* $\{S^T(t),P_i^T\}$.

Proof. Indeed, let $\{u,v\}$ be a control for γ_1. Then

$$\int_\alpha^\beta S(t)[u(t)-U(t)]\,dt+\sum_{k=1}^p P_i[v_i-V_i]=0.$$

If we set $\xi=u(t)-U(t)$, $\nu_i=v_i-V_i$, then we obtain the necessary part of the theorem.

Conversely, suppose that $u=U+\xi$, $v_i=W_i+\nu_i$. Then (42) is valid and therefore the control $\{u,v\}$ solves γ_1.

Introducing in $\mathbf{\Pi}^m[\alpha,\beta]$ the norm $\|u,v\|_m=\langle\{u,v\},\{u,v\}\rangle^{1/2}$. As in [10], one can show that $\{U,V\}$ has the smallest norm in $\mathbf{\Pi}^m[\alpha,\beta]$ among all controls solving γ_1.

REFERENCES

1. A. M. Samoilenko and N. A. Perestyuk, "Impulsive Differential Equations," World Scientific, Singapore, 1995.

2. A. D. Myshkis and A. M. Samoilenko, Systems with impulses in the prescribed moments of time, *Mat. Sb.* **74** (1967), 202–208.
3. V. Lakshmicantam, D. D. Bainov, and P. S. Simeonov, "Theory of Impulsive Differential Equations," World Scientific, Singapore, 1989.
4. M. U. Akhmetov and N. A. Perestyuk, Periodic and almost periodic solutions of strongly nonlinear impulsive systems, *J. Appl. Math. Mech.* **56** (1992), 829–837.
5. M. U. Akhmetov and N. A. Perestyuk, Periodic and almost periodic solutions of integro-differential equations with impulse actions, *Mat. Fiz. I Nelin. Mekh.* **42** (1987), 5–9.
6. M. Rama Mahana Rao, K. Srivastava Sanjay, and S. Sivasundaram, Stability of linear delay impulsive differential equations with impulsive effect, *J. Math. Anal. Appl.* **163** (1992), 47–59.
7. A. Anokhin, L. Berezansky, and E. Braverman, Stability of linear delay impulsive differential equations, *Dyn. Syst. Applic.* **4** (1995), 173–188.
8. L. V. Kantorovich and G. P. Akilov, "Functional Analysis," Pergamon Press, Oxford, 1982.
9. M. U. Akhmetov, N. A. Perestyuk, and M. A. Tleubergenova, The control of linear impulsive systems, *Ukrain. Math. Zh.* **47** (1995), 307–314.
10. N. N. Krasovsky, "Theory of Motion Control, Linear Systems," Nauka, Moscow, 1973.