ON A COMPARISON METHOD FOR PULSE SYSTEMS IN THE SPACE R^n

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A method for the study of differential equations with pulse influence on the surfaces, which was realized in [1] for a bounded domain in the phase space, is now extended to the entire space R^n. We prove theorems on the existence of integral surfaces in the critical case and justify the reduction principle for these equations.

Consider the following system of differential equations with pulse influence on the surfaces:

$$\frac{dx}{\partial t} = A(t)x + f(t, x), \quad t \neq t_i + t_i(x),$$

$$\Delta x\,|_{t=t_i+t_i(x)} = B_i x + I_i(x), \tag{1}$$

where $x \in R^n$, $t \in R$, $A(t)$ is a continuous matrix-valued function, B_i is a bounded sequence of quadratic matrices of the nth order such that $\det(E + B_i) \neq 0$, $i \in Z$ (Z is a set of integer numbers), and f is a function continuous in t. Assume that the Lipschitz condition

$$\|f(t, x) - f(t, y)\| + \|I_i(x) - I_i(y)\| \leq l\,\|x - y\| \tag{2}$$

is satisfied uniformly with respect to $t \in R$ and $i \in Z$ for all $x, y \in R^n$ and that the following relations hold:

$$\sup_t \|f(t, 0)\| + \sup_i \|I_i(0)\| = M < +\infty, \tag{3}$$

$$\sup_t \|A(t)\| = N < +\infty. \tag{4}$$

There exists a positive number $\theta \in R$ such that the inequality $t_{i+1} \geq t_i + \theta$ holds for all $i \in Z$. In addition, the surfaces of discontinuity Γ_i: $t = t_i + t_i(x)$, $i \in Z$, satisfy the following conditions:

$$t_i + t_i(x) < t_{i+1} + t_{i+1}(x), \quad |t_i| \to +\infty \quad \text{as} \quad |i| \to \infty,$$
$$t_i((E + B_i)x + I_i(x)) \leq t_i(x). \tag{5}$$

1. The S-Property in the Space R^n

Fix a real number $l_0 > 0$ and denote $h = l_0 \exp((N + l_0)\, l_0)$.

We introduce scalar functions $\nu(u)$ and $\bar{\nu}(u)$ defined on the interval $0 < u < +\infty$ and satisfying the inequalities

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$$\nu(u) < \begin{cases} 1/((N+l_0)u+M), & \text{if } M>0, \\ \min(1,\ 1/(N+l_0)u), & \text{if } M=0, \end{cases} \tag{6}$$

$$\bar{\nu}(u) < \begin{cases} 1/l_0[(N+l_0)u+M], & \text{if } M>0, \\ \min(1,\ 1/l_0(N+l_0)u), & \text{if } M=0. \end{cases} \tag{7}$$

It is easy to check that

$$\sup_{u>0}\nu(u) = \begin{cases} 1/M, & \text{if } M>0, \\ 1, & \text{if } M=0, \end{cases} \quad \text{and} \quad \sup_{u>0}\bar{\nu}(u) = \begin{cases} 1/l_0M, & \text{if } M>0, \\ 1, & \text{if } M=0. \end{cases}$$

Denote $K_h(x_0) = \{x \in R^n \mid \|x - x_0\| \le h\}$ for every $x_0 \in R^n$. Assume that the inequality

$$|t_i(x)| \le l\nu(\|x_0\|), \quad l < l_0, \quad i \in Z, \tag{8}$$

holds for $x \in K_h(x_0)$ and $x_0 \in R^n$. In addition, let the local Lipschitz condition

$$|t_i(x) - t_i(y)| \le l\bar{\nu}(\|x_0\|)\|x-y\| \tag{9}$$

be satisfied for $x, y \in K_h(x_0)$ and $x_0 \in R^n$.

The function $t = t(x)$, which satisfies conditions (8) and (9) for arbitrary $l_0, M,$ and $N,$ can be represented, for example, by the mapping $t(x) = lKe^{-\|x\|}$ with sufficiently small constant $K = K(l_0, M, N)$.

Further, by using relations (2)–(4), (7), and (9), we find that the inequality

$$\|A(t_0)x_0 + f(t_0, x_0)\|\, l\bar{\nu}(\|x_0\|) \le l/l_0 < 1 \tag{10}$$

takes place at any point (t_0, x_0) on the surface of discontinuity Γ_i $(i \in Z)$.

By virtue of the corollary of Theorem 9 [3], this inequality and the last relation in (5) imply that, for sufficiently small l, each solution of system (1) crosses any surface of discontinuity only once.

We fix $i \in Z$. Let $x_0(t) = x(t, t_i, x)$ be a solution of the system of ordinary differential equations

$$\frac{dx}{dt} = A(t)x + f(t,x). \tag{11}$$

Denote by $t = \theta_i$ the time when this solution crosses the surface Γ_i. Assume that $x_1(t) = x(t, \theta_i, (E+B_i)x_0(\theta_i) + I_i(x_0(\theta_i)))$ is also a solution of Eq. (11). Let us define a mapping $J_i(x) = x_1(t_i) - (E+B_i)x$ and construct a system of differential equations with pulse influence at fixed times, which has the form

$$\frac{dy}{dt} = A(t)y + f(t,y), \quad t \ne t_i, \quad \Delta y\,|_{t=t_i} = B_i y + J_i(y). \tag{12}$$

We say that systems (1) and (12) possess the S-property if, for any solution $x(t)$ of Eq. (1) with the discontinuity points $t = \tau_i$ $(i \in Z)$, there exists a solution $y(t)$ of system (12) satisfying the equality $x(t) = y(t)$ everywhere on R except the intervals $(t_i, \tau_i]$ (or $(\tau_i, t_i]$ if $\tau_i < t_i$).

In particular,

$$y(t_i) = x(t_i) \text{ if } \tau_i \ge t_i \quad \text{or} \quad y(t_i) = x(t_i+) \text{ if } \tau_i < t_i$$

and

$$y(\tau_i) = x(\tau_i+) \text{ if } \tau_i \ge t_i \quad \text{or} \quad y(\tau_i) = x(t_i) \text{ if } \tau_i < t_i.$$

Conversely, for each solution y of Eq. (12), there exists a solution x of system (1) satisfying all the conditions stated above. By using the method of constructing the mappings J_i, one can easily verify that Eqs. (1) and (12) possess the S-property.

Let us prove the following lemma:

Lemma 1. *Assume that system* (1) *satisfies the conditions presented above. Then there exists a real number* l_1 $(0 < l_1 < l_0)$ *such that the inequality*

$$\|J_i(x) - J_i(y)\| \le l\bar{k}(l)\|x - y\| \tag{13}$$

holds for all $x, y \in R^n$ *and* $0 < l < l_1$*; here* $\bar{k}(l)$ *is a function bounded on the set* $0 < l < l_1$.

Proof. Let (t_i, x) and (t_i, y) be points of the plane $t = t_i$ for some fixed $i \in Z$. Without loss of generality, we can assume that $\|x\| \ge \|y\|$. Let $x_0(t) = x(t, t_i, x)$ and $y_0(t) = y(t, t_i, y)$ be the solutions of Eq. (11) and let $t = \theta_i$ and $t = \eta_i$ be the times when these solutions cross the surface Γ_i, respectively. Without loss of generality, we can take $t_i \le \theta_i \le \eta_i$. Denote $x^+ = (E + B_i)x_0(\theta_i) + I_i(x_0(\theta_i))$ and $y^+ = (E + B_i)y_0(\eta_i) + I_i(y_0(\eta_i))$. Assume that $x_1(t) = x(t, \theta_i, x^+)$ and $y_1(t) = x(t, \eta_i, y^+)$ are also the solutions of system (11). Then $J_i(x) = x_1(t_i) - (E + B_i)x$ and $J_i(y) = y_1(t_i) - (E + B_i)y$. By using these expressions, we obtain

$$\begin{aligned}\|J_i(x) - J_i(y)\| \le{}& \|E + B_i\|\int_{t_i}^{\theta_i} (\|A(u)\| + l)\|x_0(u) - y_0(u)\|\,du \\ &+ \|E + B_i\|\int_{\theta_i}^{\eta_i} ((\|A(u)\| + l)\|y_0(u)\| + M)\,du \\ &+ l\left\{\|x - y\| + \|E + B_i\|\int_{t_i}^{\theta_i} (\|A(u)\| + l)\|x_0(u) - y_0(u)\|\,du + \int_{\theta_i}^{\eta_i} ((\|A(u)\| + l)\|y_0(u)\| + M)\,du\right\} \\ &+ \int_{\eta_i}^{\theta_i} ((\|A(u)\| + l)\|y_1(u)\| + M)\,du + \int_{\theta_i}^{\eta_i} (\|A(u)\| + l)\|x_1(u) - y_1(u)\|\,du.\end{aligned} \tag{14}$$

Let us find the upper bound of the right-hand side of (14). Since it contains integrals of the same type, it suffices to determine the upper bound of the range of the values for the integrals

$$\alpha = \int_{t_i}^{\theta_i} (\|A(u)\| + l)\|x_0(u) - y_0(u)\|\,du \quad \text{and} \quad \beta = \int_{\theta_i}^{\eta_i} ((\|A(u)\| + l)\|y_0(u)\| + M)\,du.$$

First, by virtue of the conditions of Lemma 1, we find

$$\| x_0(t) - x) \| \le h, \quad \| y_0(t) - y) \| \le h, \quad \| x_1(t) - x^+) \| \le h, \quad \| y_1(t) - y^+) \| \le h. \tag{15}$$

We only establish the first relation in (15), since the others can be proved similarly. Assume the contrary. Then there exists a point $\bar{t} \in (t_i, \theta_i]$ such that the relations $\| x_0(\bar{t}) - x \| = h$ and $\| x_0(t) - x \| < h$ hold for all $t \in (t_i, \bar{t})$. Hence, taking the inequality

$$\| x_0(t) - x \| \le \int_{t_i}^{t} (\| A(u) \| \, \| x \| + \| f(u, x) \|) \, du + \int_{t_i}^{t} (\| A(u) \| + l) \, \| x_0(u) - x \| \, du$$

into account and employing the Gronwall–Bellman lemma, we obtain

$$\| x_0(t) - x \| \le l_0 \nu (\| x \|) ((N + l_0) \| x \| + M) e^{(N + l_0) l_0}.$$

This and inequality (6) imply that the relation $\| x_0(t) - x \| \le h$ holds for all $t \in (t_i, \bar{t}]$, but this contradicts the equality $\| x_0(\bar{t}) - x \| = h$. Hence, inequality (15) is true.

It follows from (15) that $\theta_i - t_i \le l \nu (\| x \|)$ and $\eta_i - t_i \le l \nu (\| y \|)$. To estimate $\eta_i - \theta_i$, it is necessary to consider the following two cases:

1. Let $y_0(\eta_i) \in K_h(x_0(\theta_i))$. Then

$$\eta_i - \theta_i = t_i (y_0(\eta_i)) - t_i (x_0(\theta_i)) \le l \bar{\nu} (\| y_0(\eta_i) \|) \left(\int_{\theta_i}^{\eta_i} f(u, y_0(u)) \, du + \| y_0(\theta_i) - x_0(\theta_i) \| \right). \tag{16}$$

Since the equality

$$y_0(t) - x_0(t) = y - x + \int_{t_i}^{t} (A(u)(y_0(u) - x_0(u)) + f(u, y_0(u)) - f(u, x_0(u))) \, du$$

holds on the segment $[t_i, \theta_i]$, by employing the Gronwall–Bellman lemma, we obtain

$$\| y_0(\theta_i) - x_0(\theta_i) \| \le \| y - x \| e^{(N + l) l}. \tag{17}$$

The validity of the relation

$$\| y_0(t) \| \le (\| y \| + M l \nu (\| y \|)) e^{(N + l) l}$$

can be verified in exactly the same way. Thus, we have

$$\left\| \int_{\theta_i}^{\eta_i} f(u, y_0(u)) \, du \right\| \le (\eta_i - \theta_i)[M + l(\| y_0(\eta_i) \| + M l \nu (\| y_0(\eta_i) \|) \, e^{(N + l) l}]. \tag{18}$$

Let us estimate the right-hand side of (16) by using relations (17) and (18). We find that

$$\eta_i - \theta_i \le \frac{l \bar{\nu} (\| y_0(\eta_i) \|) e^{(N + l) l} \| x - y \|}{1 - l \bar{\nu} (\| y_0(\eta_i) \|) \left[M + l (\| y_0(\eta_i) \| + M l \nu (\| y_0(\eta_i) \|)) e^{(N + l) l} \right]} \tag{19}$$

holds. In order that the last expression be meaningful for sufficiently small l, it is sufficient that the function

$$F(u) = \bar{\nu}(u)\,[M + l(u + Ml\nu(u)\,e^{(N+l)l}]$$

be bounded, and this is true by virtue of inequalities (6) and (7).

It follows from (19) that there exists a bounded function $k_1(l)$ such that the relation

$$\eta_i - \theta_i \le lk_1(l)\bar{\nu}(\|y_0(\eta_i)\|)\,\|x - y\| \tag{20}$$

holds for sufficiently small l.

2. Let $y_0(\eta_i) \overline{\in} K_h(x_0(\theta_i))$. First, we show that this condition is satisfied for sufficiently small l only in the case where $\|x-y\| \le h/2$. In fact,

$$\|y_0(\eta_i) - y\| = \left\| \int_{t_i}^{\eta_i} f(u, y_0(u))du \right\| \le l\bar{\nu}(\|y\|)[M + l(\|y\| + Ml\nu(\|y\|))e^{(N+l)l}].$$

This and the boundedness of the function $F(u)$ imply that the inequality $\|y_0(\eta_i) - y\| \le h/4$ holds for sufficiently small l. By analogy, if l is sufficiently small, then the relation $\|x_0(\theta_i) - x\| \le h/4$ holds. By using these inequalities, we obtain

$$\|x-y\| \ge \|y_0(\eta_i) - x_0(\theta_i)\| - \|y_0(\eta_i) - y\| - \|x_0(\theta_i) - x\| \ge h/2.$$

Hence, we have

$$\eta_i - \theta_i = t_i(y_0(\eta_i)) - t_i(x_0(\theta_i)) \le l\nu(\|y_0(\eta_i)\|) \le l\nu(\|y_0(\eta_i)\|)\,\frac{2}{h}\,\|x - y\|. \tag{21}$$

Assume that $k(l) = \max(2/h, k_1)$ and $\bar{\bar{\nu}} = \max(\nu(u), \bar{\nu}(u))$. Then relations (20) and (21) yield

$$|\eta_i - \theta_i| \le lk(l)\,\bar{\bar{\nu}}(\|y_0(\eta_i)\|)\,\|x - y\|. \tag{22}$$

We now proceed to estimating α and β. We have

$$\alpha \le (\theta_i - t_i)(N + l)e^{(N+l)l}\|x-y\| \le l\nu(\|x\|)\,(N + l)\,e^{(N+l)l}\|x-y\|. \tag{23}$$

Thus, by applying (22), we get

$$\beta \le (\eta_i - \theta_i)(M + (N + l_0)(\|y_0(\eta_i)\| + Ml_0))\,e^{(N+l_0)l_0}$$

$$\le lk(l)\bar{\bar{\nu}}(\|y_0(\eta_i)\|)(M + (N + l_0)(\|y_0(\eta_i)\| + Ml_0))\exp((N + l_0)l_0))\,\|x-y\|,$$

whence, by virtue of the uniform boundedness of the function

$$\bar{\bar{\nu}}(u)\,(M + (N + l_0)(u + Ml_0))\exp((N + l_0)l_0))$$

in $u > 0$, we find that $\beta \le l k_2(l) \| x - y \|$, where k_2 is a bounded function.

By analyzing the estimates obtained for α and β, we conclude that similar relations are valid for all the terms on the right-hand side of (14). This implies the validity of relation (13). Lemma 1 is proved.

2. Integral Surfaces for Systems with Constant Coefficients of Some Variables

Let us decompose the space R^n into the direct product $R^m \times R^{n-m}$ and define a norm in it as follows:

If $w = (x, y)$, where $x \in R^m$ and $y \in R^{n-m}$, then $\| w \| = \| x \| + \| y \|$, where $\| x \|$ and $\| y \|$ are the Euclidean norms of the vectors x and y.

On the set $R^1 \times R^n$, we consider the following system of differential equations with pulse influence on the surfaces:

$$\frac{dx}{dt} = A(t)x + f_1(t, w),$$

$$\frac{dy}{dt} = \mathbb{C} y + f_2(t, w), \qquad t \neq t_i + t_i(w), \tag{24}$$

$$\Delta x \big|_{t = t_i + t_i(w)} = B_i x + I_i^1(w), \qquad \Delta y \big|_{t = t_i + t_i(w)} = I_i^2(w),$$

where A is an $m \times m$ matrix continuous in t, $\mathbb{C}$ and B_i $(i \in Z)$ are constant quadratic matrices, and $E + B_i$ $(i \in Z)$ are nonsingular matrices. Further, denote $f = (f_1, f_2)$ and $I_i = (I_i^1, I_i^2)$ and assume that the inequality

$$\| f(t, w) - f(t, \overline{w}) \| + \| I_i(w) - I_i(\overline{w}) \| \le l \| w - \overline{w} \| \tag{25}$$

holds uniformly in $t \in R$ and $i \in Z$ for all $w, \overline{w} \in R^n$. Let

$$f(t, 0) = I_i(0) = 0, \tag{26}$$

$$\sup_t \| A \| + \sup_i \| B \| + \| \mathbb{C} \| = N < +\infty. \tag{27}$$

We also assume that the real numbers t_i $(i \in Z)$ satisfy the condition

$$0 < \theta_1 \le t_{i+1} - t_i \le \theta_2 < +\infty \tag{28}$$

and that the functions $t_i(x)$ satisfy relations (8) and (9) and the inequalities

$$t_i + t_i(w) < t_{i+1} + t_{i+1}(w) \quad \text{and} \quad t_i\big(w + (B_i x, 0) + I_i(w)\big) \le t_i(w). \tag{29}$$

It is easy to see that Eq. (24) is a system of type (1). Therefore, by using the results obtained in Section 1, we can establish some relations for system (24) by analogy with the corresponding results for Eq. (1).

First, we note that conditions (8), (9), and (29) eliminate "beating" of the solutions to system (24) against the discontinuity surfaces for sufficiently small l.

By setting $z = (\xi, \eta)$, we construct the equations

$$\begin{aligned}\frac{d\xi}{dt} &= A(t)\xi + f_1(t,z), \quad \frac{d\eta}{dt} = C\eta + f_2(t,z), \quad t \neq t_i,\\ \Delta\xi|_{t=t_i} &= B_i\xi + J_i^1(z), \quad \Delta\eta|_{t=t_i} = J_i^2(z),\end{aligned} \tag{30}$$

by analogy with system (12).

System (30) and Eq. (24) possess the S-property in the entire space R^n. Denote $J_i = (J_i^1, J_i^2)$. For the vectors J_i $(i \in Z)$, the relations

$$J_i(0) = 0, \quad \|J_i(z) - J_i(\bar{z})\| \le l\bar{k}(l)\|z - \bar{z}\|$$

hold uniformly for all $z, \bar{z} \in R^n$; here, $\bar{k}$ is a bounded function. Further, assume that

$$\operatorname{Re}\lambda_j(\mathbb{C}) \ge 0, \quad j = \overline{m+1, n}, \tag{31}$$

and the matricant $X(t, u)$ of the linear homogeneous system $dx/dt = A(t)x$, $t \neq t_i$, $\Delta x|_{t=t_i} = B_i x$, corresponding to Eq. (30), satisfies the estimate

$$\|X(t,u)\| \le ae^{-\lambda(t-u)}, \quad t \ge u, \tag{32}$$

where a and λ are positive constants.

The problem of the existence of integral surfaces for the equations of type (30) has already been investigated in [4]. Therefore, we only formulate Lemmas 2–4.

It is well known that inequality (31) implies the relation $\|\exp(\mathbb{C}t)\| \le a(1 + |t|^k)$ $(t \le 0)$, where k, $0 \le k \le n - m$, is an integer and the coefficient a is taken, without loss of generality, to be equal to the coefficient in (32).

Let us choose an arbitrary σ from the interval $(0, \lambda)$ and denote

$$\nu_1 = \int_0^\infty (1 + t^k)e^{-\sigma t}\,dt \quad \text{and} \quad \nu_2 = \sup_t \sum_{t < t_i} (1 + (t_i - t)^k)\, e^{-\sigma(t_i - t)}.$$

It can be shown that ν_1 and ν_2 are finite numbers.

Fix a positive number $\varepsilon \in R$ and denote

$$\mu_1 = la(a + \varepsilon)((\lambda - \sigma)^{-1} + \nu_1 + \bar{k}(l)(1 - e^{-(\lambda-\sigma)\theta_1})^{-1} + \nu_2),$$

$$\mu_2 = la(\lambda^{-1} + \nu_1 + \bar{k}(l)((1 - e^{-\lambda\theta_1})^{-1} + \nu_2)).$$

Lemma 2. *Assume that system* (30) *satisfies the conditions given above. Let* $z(t)$, $z(t_0) = (\xi_0, \eta_0)$, *be the solution of this equation satisfying the relation* $\|z(t)\| \le c_z \exp(-\sigma(t - t_0))$, $t \ge t_0$, *where* c_z *is a positive constant. Then the function* $z(t)$ *satisfies the following system of integral equations:*

$$\begin{aligned}\xi &= X(t, t_0)\xi_0 + \int_{t_0}^{t} X(t,u) f_1(u,z)\,du + \sum_{t_0 \le t_i < t} X(t, t_i) J_i^1(z),\\ \eta &= -\int_t^\infty e^{\mathbb{C}(t-u)} f_2(u,z)\,du - \sum_{t < t_i} e^{\mathbb{C}(t-t_i)} J_i^2(z),\end{aligned} \tag{33}$$

and

$$\eta_0 = -\int_{t_0}^{\infty} e^{\mathbb{C}(t_0-u)} f_2(u, z(u))\, du - \sum_{t_0<t_i} e^{\mathbb{C}(t_0-t_i)} J_i^2(z(t_i)). \tag{34}$$

If $\mu_2 < 1$, then the vector η_0 is determined by expression (34) uniquely for a given ξ_0. Assume that

$$F(\xi_0, t_0) = -\int_{t_0}^{\infty} e^{\mathbb{C}(t_0-u)} f_2(u, z(u))\, du - \sum_{t_0<t_i} e^{\mathbb{C}(t_0-t_i)} J_i^2(z(t_i)).$$

By using Lemma 2, we can prove the following lemma:

Lemma 3. *Assume that the conditions imposed on system* (30) *are satisfied and a constant* l *is chosen sufficiently small to guarantee the validity of the inequalities* $\mu_1 < \varepsilon$ *and* $\mu_2 < 1$. *Then the equation* $\eta = F(\xi, t)$ *determines an m-dimensional integral manifold* Ψ_+ *of system* (30). *Every solution* $z(t) = z(t, \xi_0)$, $z(t_0) = z(\xi_0, \eta_0)$, *whose integral curve lies on* Ψ_+, *satisfies the inequality*

$$\| z(t) \| \leq (a+\varepsilon) \| \xi_0 \| \exp(-\sigma(t-t_0)).$$

Moreover, the function F *satisfies the relations*

$$F(0, t) = 0 \quad \text{and} \quad \| F(\xi_1, t) - F(\xi_2, t) \| \leq 2la^2 \| \xi_1 - \xi_2 \|$$

uniformly in $t \in R$.

All the solutions $z(t, \xi_0)$ *continuously depend on* ξ_0 *in the* B*-topology* [1].

By virtue of the S-property, Lemma 3 implies the following theorem:

Theorem 1. *Assume that relations* (8), (9), (25)–(29), *and* (32) *hold,* $\mu_1 < \varepsilon$, *and* $\mu_2 < 1$. *Then there exists an* m*-dimensional integral surface* Φ_+ *of system* (24) *such that each solution* $x(t, \xi)$, *whose integral curve belongs to* Φ_+ *for* $t \geq t_0$, *satisfies the inequality*

$$\| x(t, \xi) \| \leq d(\| \xi \|)\, e^{-\sigma(t-t_0)}, \qquad d(u) = O(u).$$

If θ_i *are the discontinuity points of the solution* $x(t, \xi)$, *then* $|\theta_i - t_i - t_i(0)| \to 0$ *as* $i \to +\infty$.

If Eq. (24) *is periodic in* t *with period* ω, *then the integral set* Φ_+ *is also* ω*-periodic.*

Consider the problem of the existence of the integral surface of Eq. (24) connected with the matrix $\mathbb{C}$.

First, we formulate the following lemma:

Lemma 4. *Assume that system* (30) *satisfies the above-mentioned conditions. Then, for sufficiently small constant* l *in the Lipschitz condition, there exists an* $(n-m)$*-dimensional integral surface* Ψ_- *of this system defined by the equation* $\xi = Q(\eta, t)$, *where* Q *is an* m*-dimensional function piecewise continuous in* t *and satisfying the conditions*

$$Q(0, t) = 0 \quad \text{and} \quad \| Q(\eta_1, t) - Q(\eta_2, t) \| \leq \kappa l \tilde{k}(l) \| \eta_1 - \eta_2 \|,$$

where κ *is a positive constant depending only on the matrices* A, B_i, *and* $\mathfrak{C}$, *and* $\tilde{k}$ *is a bounded function.*

Every solution $z(t)$ *located on* Ψ_- *satisfies the estimate* $\|z(t)\| \leq c_z \exp(-\beta t)$, $t \leq 0$, *where* $c_z > 0$ *and* $\beta > 0$ *are constants; moreover,* β *is arbitrarily small for sufficiently small* l.

Lemma 4 implies the validity of the following theorem:

Theorem 2. *Assume that relations* (8), (9), (25)–(29), (32), *and* (35) *hold for system* (24). *Then, for sufficiently small* l, *this system possesses an* $(n-m)$*-dimensional integral surface* Φ_-, *on which all the solutions satisfy the estimate*

$$\|w(t)\| \leq c_z \exp(-\beta t), \quad t \leq 0,$$

where $\beta > 0$ *is an arbitrarily small constant for sufficiently small* l.

3. The Reducing Principle for Weakly Linear Pulse Systems

We proceed with the study of system (24) and assume that the conditions of Theorems 1 and 2 are satisfied. Hence, the surfaces Φ_- and Φ_+ exist. Denote by $x = \overline{Q}(y,t)$ the equation determining the integral surface Φ_-. Then Eq. (24), on the surface Φ_-, can be written in the form

$$\frac{dy}{dt} = \mathfrak{C}y + f_2(t, \overline{Q}(y,t), y), \quad t \neq t_i + t_i(\overline{Q}(y,t), y),$$

$$\Delta y\,|_{t=t_i+t_i(\overline{Q}(y,t),\,y)} = I_i^2(\overline{Q}(y,t), y). \tag{35}$$

By using the reducing principle developed for ordinary differential equations [5], we can study the problem of stability of the trivial solution to Eq. (24) dependently on the stability of the trivial solution to system (35). By virtue of the S-property established for systems (24) and (30), it suffices to consider Eq. (30). Thus, our principal goal is to establish the direct dependence between the stability of the trivial solution to Eq. (30) and the stability of the trivial solution to the system

$$\begin{aligned} \frac{d\eta}{dt} &= \mathfrak{C}\eta + f_2(t, Q(\eta,t), \eta), \quad t \neq t_i, \\ \Delta\eta\,|_{t=t_i} &= J_i^2(Q(\eta, t_i), \eta). \end{aligned} \tag{36}$$

(Equation (30) reduces to this system on the surface Ψ_-.)

Before studying this problem, let us investigate the stability of the surface Ψ_-. We introduce the necessary definitions.

Consider the following system of differential equations with pulse influence on the surfaces:

$$\frac{dx}{dt} = P(t,x), \quad t \neq t_i(x), \quad \Delta x\,|_{t=t_i(x)} = W_i(x). \tag{37}$$

Let $\mathfrak{A}$ be the integral surface of this equation and $\tau \in R$. Denote $\mathfrak{A}_\tau = \{x \in R^n \,|\, (\tau, x) \in \mathfrak{A}\}$.

***Definition* 1.** *A solution* $x(t)$ *of Eq.* (37) *lies in the* ε*-neighborhood of the set* $\mathfrak{A}$ *on the interval* J *if*

$x(t)$ is defined on this interval and, for all $\tau \in J$ lying outside the ε-neighborhoods of the discontinuity points of the solution $x(t)$, the point $x(\tau)$ belongs to the ε-neighborhood of the set $\mathfrak{A}_\tau$, i.e., the following inequality holds: $\inf\limits_{x \in \mathfrak{A}_\tau} \| x(\tau) - x \| < \varepsilon$.

***Definition* 2.** *The integral surface $\mathfrak{A}$ is B-stable if, for any real number $\varepsilon > 0$, one can find a real number $\delta > 0$ such that any solution $x(t) = x(t, t_0, x_0)$, for which $t = t_0$ is not a discontinuity point and x_0 belongs to the δ-neighborhood of the set $\mathfrak{A}_{t_0}$, is situated in the ε-neighborhood of the set $\mathfrak{A}$ for $J = \{ t \in R \mid t \geq t_0 \}$.*

***Definition* 3.** *The integral surface $\mathfrak{A}$ is B-asymptotically stable if there exists a positive number $\Delta \in R$ such that, for every real number $\varepsilon > 0$ and a solution $x(t) = x(t, t_0, x_0)$, for which $t = t_0$ is not a discontinuity point and x_0 belongs to the Δ-neighborhood of the set $\mathfrak{A}_{t_0}$, one can find a real number $\theta > t_0$ such that the solution $x(t)$ belongs to the ε-neighborhood of the set $\mathfrak{A}$ for $J = \{ t \in R \mid t \geq \theta \}$.*

***Definition* 4.** *The integral surface $\mathfrak{A}$ is called totally B-stable if the number $\Delta > 0$ in Definition 3 can be chosen arbitrarily.*

Assume that $z(t) = (\xi, \eta)$, $z(t_0) = (\xi_0, \eta_0)$, is an arbitrary solution of system (30) and that $z(t, \bar{\eta}) = (\xi(t, \bar{\eta}), \eta(t, \bar{\eta}))$ is a solution of this system satisfying the initial condition $\xi(t_0, \bar{\eta}) = Q(\bar{\eta}, t_0)$, $\eta(t_0, \bar{\eta}) = \bar{\eta}$ and, therefore, lying on the surface Ψ_-. In Eq. (30), we change the variables as follows: $x = \xi - \xi(t, \bar{\eta})$ and $y = \eta - \eta(t, \bar{\eta})$. Denote $w = (x, y)$. As a result, we obtain the following system of equations for w:

$$\begin{gathered} \frac{dx}{dt} = A(t)x + F_1(t, w, \bar{\eta}), \qquad \frac{dy}{dt} = \mathfrak{C}y + F_2(t, w, \bar{\eta}), \qquad t \neq t_i, \\ \Delta x\,|_{t=t_i} = B_i x + V_i^1(w, \bar{\eta}), \qquad \Delta y\,|_{t=t_i} = V_i^2(w, \bar{\eta}), \end{gathered} \tag{38}$$

where the functions $F = (F_1, F_2)$ and $V_i = (V_i^1, V_i^2)$ satisfy the relations

$$\| F(t, w, \bar{\eta}) - F(t, \bar{w}, \bar{\eta}) \| \leq l \| w - \bar{w} \| \quad \text{and} \quad \| V_i(w, \bar{\eta}) - V_i(\bar{w}, \bar{\eta}) \| \leq l k(l) \| w - \bar{w} \|$$

uniformly in all $t \in R$, $t \in Z$, and $w, \bar{w} \in R^n$.

According to Lemma 3, system (38) admits a piecewise continuous integral surface $\tilde{\Psi}_+$ determined by equation of the form $y = \tilde{F}(x, t, \bar{\eta})$, where the function $\tilde{F}$ is continuous in the parameter $\bar{\eta}$ and satisfies the relations

$$\tilde{F}(0, t, \bar{\eta}) = 0, \quad t \in R, \qquad \| \tilde{F}(x, t, \bar{\eta}) - \tilde{F}(\bar{x}, t, \bar{\eta}) \| \leq \kappa l \| x - \bar{x} \|,$$

where κ is a constant satisfying the inequality $\kappa \geq 2la^2(\nu_1 + \nu_2)$.

On the basis of the properties of the functions Q and $\tilde{F}$, we now establish that the surface Ψ_- is totally stable.

Lemma 5. *For each solution $z(t)$ of system* (30), *one can find a solution $z(t, \bar{\eta})$ of the same system such that*

$$\|\xi(t) - \xi(t, \bar{\eta})\| \le 2a\|\xi_0 - \xi(t_0, \bar{\eta})\| e^{-\sigma(t-t_0)},$$

$$\|\eta(t) - \eta(t, \bar{\eta})\| \le 2a\kappa l\|\xi_0 - \xi(t_0, \bar{\eta})\| e^{-\sigma(t-t_0)}, \qquad t \ge t_0.$$

The proof of Lemma 5 mostly coincides with the proof of the theorem in [4].

In exactly the same way (to within constants), one can establish the validity of the next lemma by using Lemma 5 and the argument in [4].

Lemma 6. *Suppose that the conditions imposed on Eq.* (30) *are satisfied and the constant* l *in* (8), (9), *and* (25) *is such that* $l \to 0$ *as* $\|z\| + \|\bar{z}\| \to 0$. *Then the trivial solution of this system is stable, asymptotically stable, or unstable provided that the trivial solution of the corresponding system* (36) *is stable, asymptotically stable, or unstable, respectively.*

By virtue of the S-property established for systems (24) and (30), we can formulate Theorems 3 and 4, which follow from Lemmas 5 and 6, respectively.

Theorem 3. *For sufficiently small* l, *the integral surface* Φ_- *of system* (24) *is totally* B*-stable.*

Theorem 4. *Assume that the system of equations* (24) *is critical, conditions* (31) *and* (32) *are satisfied, and the constant* l *in relations* (8), (9), *and* (25) *is such that* $l \to 0$ *as* $\|w\| + \|\bar{w}\| \to 0$.

Then the trivial solution of system (24) *is stable, asymptotically stable, or unstable if and only if the trivial solution of system* (35) *is stable, asymptotically stable, or unstable, respectively.*

One can easily verify that Theorem 4 is local. Therefore, it is also valid in the case where the discontinuity surfaces satisfy the weaker conditions given in [1].

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