PERIODIC SOLUTIONS OF STRONGLY NONLINEAR SYSTEMS WITH NONCLASSICAL RIGHT-HAND SIDE IN THE CASE OF A FAMILY OF GENERATING SOLUTIONS

M. U. Akhmetov

UDC 517.911

The problem of the existence of periodic solutions to differential equations with pulse effects on the surfaces and to differential equations with discontinuous right-hand sides close to arbitrary nonlinear ones is studied. The existence of a family of periodic solutions to generating equations is assumed.

Equations of the type considered below have been investigated in [1–6]. The theorems presented in this paper generalize the statements in [7]. We use the results obtained in [8–10].

1. Pulse Systems

Let Ω_x be a domain in R^n having a compact closure, and let μ_0 be a fixed positive number. On the set

$$\Omega = \{(x, t, i, \mu \mid x \in \Omega_x, \quad -\infty < t < \infty, \quad i = 0, \pm 1, \ldots, \quad -\mu_0 < \mu < \mu_0\},$$

we consider the following system of differential equations with pulse effects on the surfaces

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) + \mu g(t, x, \mu), \quad t \neq t_i(x) + \mu\tau_i(x, \mu), \\ \Delta x \big|_{t = t_i(x) + \mu\tau_i(x, \mu)} &= I_i(x) + \mu W_i(x, \mu), \end{aligned} \tag{1}$$

where the functions I_i, t_i, W_i, and τ_i have continuous partial derivatives of second order with respect to the variables μ, x_j, $j = \overline{1, n}$, $f \in C^{(0,2)}(\Omega) \cap C^{(1,2)}(\Omega_0)$, $g \in C^{(0,1,1)}(\Omega) \cap C^{(1,2,2)}(\Omega_0)$, where Ω_0 is the union of certain neighborhoods of the surfaces $t = t_i(x)$, $i = 0, \pm 1, \ldots$.

In what follows, for simplicity, the vectors x, f, and I and their derivatives are considered to be column vectors; the derivatives of the functions t_i are considered to be row vectors. We assume that there exist a real number $\omega > 0$ and an integer $p > 0$ for which following equalities hold uniformly in the domain Ω :

$$\begin{aligned} &f(t + \omega, x) = f(t, x), \quad g(t + \omega, x, \mu) = g(t, x, \mu), \\ &I_{i+p} = I_i, \quad W_{i+p} = W_i, \quad t_{i+p} = t_i + \omega, \quad \tau_{i+p} = \tau_i. \end{aligned} \tag{2}$$

Suppose that the generating system

$$\frac{dx}{dt} = f(t, x), \quad t \neq t_i(x), \quad \Delta x \big|_{t = t_i(x)} = I_i(x) \tag{3}$$

has an m-parametric family $(m < n)$ of solutions $\varphi(t, \alpha)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, periodic with period ω and such

---

Kiev University. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 45, No. 2, pp. 202–208, February, 1993. Original article submitted April 4, 1991.

that at least one of the minors of the matrix $\varphi'_\alpha(0, \alpha_0)$ of order m differs from zero for some fixed $\alpha = \alpha_0$. The solution $\varphi(t, \alpha_0)$ is called generating.

Suppose that the generating solution intersects the surfaces $t = t_i(x)$ at times $\theta_i = \theta_i(\alpha_0)$, $0 < \theta_1 < \theta_2 < \ldots < \theta_p < \omega$ and that the following equality holds:

$$1 - \frac{\partial t_i(\varphi(\theta_i, \alpha_0))}{\partial x} f(\theta_i, \varphi(\theta_i, \alpha_0)) \neq 0. \tag{4}$$

Then it is easy to verify that, by virtue of the continuous dependence of solutions on initial conditions and parameters [9], all the solutions of system (1) starting from a sufficiently small neighborhood of the point $(0, \varphi(0, \alpha_0))$ intersect each discontinuity surface just once if $|\mu|$ is small.

We fix i. Let $x_0(t)$ be a solution of the equation

$$\frac{dx}{dt} = f(t, x) + \mu g(t, x, \mu) \tag{5}$$

with initial conditions $x_0(\theta_i) = x$, let ξ_i be a solution of the equation $\xi = t_i(x_0(\xi)) + \mu\tau_i(x_0(\xi), \mu)$, and let $x_1(t)$ be a solution of Eq. (5) with initial conditions $x_1(\xi_i) = x_0(\xi_i) + I_i(x_0(\xi_i)) + \mu W_i(x_0(\xi_i), \mu)$. Suppose that the solutions x_0 and x_1 exist. Let us construct the mapping $\mathcal{J}_i : (x, \mu, \alpha) \to x_1(\theta_i)$. We can check that the sequence $\mathcal{J}_i$ has period p. In addition, by using the Hadamard lemma and Lemma 1.5 in [8], we can show that the representation $J_i(x, \mu, \alpha) = Q_i(x, \alpha) + \mu H_i(x, \mu, \alpha)$ holds, where the function $Q_i(x, \alpha) = J_i(x, 0, \alpha)$ has continuous second-order derivatives with respect to each variable x_j, $j = \overline{1, n}$, α_k, $k = \overline{1, m}$; H_i is a continuously differentiable function, and equalities $H_{i+p} = H_i$ and $Q_{i+p} = Q_i$ hold uniformly in x, μ, and i.

Let us construct the following system of differential equations with pulse effects at fixed times:

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) + \mu g(t, y, \mu), \quad t \neq \theta_i, \\ \Delta y \big|_{t=\theta_i} &= Q_i(y, \alpha) + \mu H_i(y, \mu, \alpha). \end{aligned} \tag{6}$$

The following property takes place [10]:

(A) If $x(t)$ and $y(t)$ are solutions of Eqs. (1) and (6), respectively, which have the same initial conditions and the same domain of definition, and ξ_i are points of discontinuity of the solution $x(t)$, then for any i, we have $x(\theta_i) = y(\theta_i)$ when $\xi_i \geq \theta_i$ or $x(\theta_i) = y(\theta_i +)$ when $\xi_i < \theta_i$.

In what follows, the values of functions and their derivatives at the points $(\theta_i, \varphi(\theta_i, \alpha_0), 0)$ and $(\theta_i, \varphi(\theta_i +, \alpha_0), 0)$ will be written without indication of values of the arguments, and the first case will be distinguished from the second one by the superscript +.

We define products of vectors and matrices according to the rule of multiplication of rectangular matrices.

The system of equations in variations with respect to the solution $\varphi(t, \alpha_0)$ has the form

$$\frac{du}{dt} = A(t, \alpha_0)u, \quad t \neq \theta_i, \quad \Delta u\big|_{t=\theta_i} = P_i(\alpha_0)u, \tag{7}$$

where

$$A(t, \alpha_0) = \frac{\partial f(t, \varphi(t, \alpha_0))}{\partial x},$$

$$P_i(\alpha_0) = (f - f^+)\frac{\partial t_i/\partial x}{1-(\partial t_i/\partial x)f} + \frac{\partial I_i}{\partial x}\left(E + f\frac{\partial t_i/\partial x}{1-(\partial t_i/\partial x)f}\right) f, \quad \det(E + P_i(\alpha_0)) \neq 0;$$

E is the unit $n \times n$ matrix.

It follows from the assumption concerning the minors of the matrix $\varphi'_\alpha(0, \alpha_0)$ that Eq. (7) admits m linearly independent ω-periodic solutions $\partial\varphi(t, \alpha_0)/\partial\alpha_k$, $k = \overline{1, m}$. This implies, in turn, that the system conjugate with respect to (7) possesses m ω-periodic solutions ψ_k, $k = \overline{1, m}$, as well. Let us represent these solutions in the form of a matrix $\Psi(t, \alpha_0)$ of dimensionality $m \times n$.

Consider the linear inhomogeneous system

$$\begin{aligned} \frac{dx}{dt} &= A(t, \alpha_0)x + g(t, \varphi(t, \alpha_0), 0), \quad t \neq \theta_i, \\ \Delta x\big|_{t=\theta_i} &= P_i(\alpha_0)x + H_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0). \end{aligned} \tag{8}$$

As is known [3], this equation may have an ω-periodic solution if and only if the equality $\nu(\alpha_0) = 0$ takes place, where

$$\nu(\alpha) = \int_0^\omega \Psi(t,\alpha)g(t,\varphi(t,\alpha),0)\,dt + \sum_{i=1}^{p} \Psi(\theta_i,\alpha)H_i(\varphi(\theta_i,\alpha),0,\alpha).$$

In this case, an ω-periodic solution of system (8) has the form

$$\delta(t, \gamma) = \gamma_1\frac{\partial\varphi}{\partial\alpha_1} + \ldots + \gamma_m\frac{\partial\varphi}{\partial\alpha_m} + \zeta(t),$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is a real-valued vector, ζ is a special ω-periodic solution of system (8).

Theorem 1. *Assume that the above-mentioned conditions for system* (1) *are satisfied. Let also* $\alpha = \alpha_0$ *be a root of the equation* $\sigma(\alpha) = 0$, *where*

$$\sigma(\alpha) = \int_0^\omega \Psi(t,\alpha)g(t,\varphi(t,\alpha),0)dt + \sum_{i=1}^{p} \Psi(\theta_i,\alpha)\left\{\left[\left(E+\frac{\partial I_i}{\partial x}\right)f - f^+\right]\frac{\tau_i}{1-(\partial t_i/\partial x)f} + W_i\right\}$$

satisfies the inequality

$$\det\left(\frac{\partial\sigma(\alpha_0)}{\partial\alpha}\right) \neq 0.$$

Then, for sufficiently small $|\mu|$, *system* (1) *admits an* ω*-periodic solution of the form* $x = \varphi(t, \alpha_0) + \mu\xi$, *where the function* ξ *tends in B-topology to one of the solutions* $\delta(t, \gamma)$ *as* $\mu \to 0$; *the vector* γ *in this solution is determined from the nondegenerate linear system.*

Proof. The initial value of the ω-periodic solution $x(t, \eta, \mu, \alpha)$, $x(0, \eta, \mu, \alpha) = \eta$ of the system (6) is completely defined by the equation

$$D(\eta, \mu, \alpha) \equiv x(\omega, \eta, \mu, \alpha) - \eta = 0. \tag{9}$$

Arguing as in the proof of the corresponding statement in [7], analyzing Eq. (9) and using the theorems on linear pulse systems [3] and on the dependence of solutions on initial conditions, we find that if the solution $\alpha = \alpha_0$ of the equation $\nu(\alpha) = 0$ is such that

$$\det\left(\frac{\partial \nu(\alpha_0)}{\partial \alpha}\right) \neq 0,$$

then, for sufficiently small $|\mu|$, system (6) admits a unique ω-periodic solution approaching the generating solution as $\mu \to 0$.

Further, by using the relation

$$H_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0) = \frac{\partial \mathcal{J}_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0)}{\partial \mu},$$

we obtain

$$H_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0) = \left[\left(E + \frac{\partial I_i}{\partial x}\right) f - f^{+}\right] \frac{\tau_i}{1 - (\partial t_i / \partial x) f} + W_i,$$

i.e., $\nu(\alpha_0) = \sigma(\alpha_0)$.

The statement concerning the existence of an ω-periodic solution of system (1) follows from the property (A).

We now change the variables $y = \varphi(t, \alpha_0) + \mu z$ in Eq. (6) and pass to the system

$$\begin{aligned} \frac{dz}{dt} &= A(t, \alpha_0) z + h(t, \alpha_0) + \mu F(t, z, \mu, \alpha_0), \quad t \neq \theta_i, \\ \Delta z|_{t=\theta_i} &= P_i(\alpha_0) z + \vartheta_i(\alpha_0) + \mu V_i(z, \mu, \alpha_0), \end{aligned} \tag{10}$$

where

$$h(t, \alpha_0) = g(t, \varphi(t, \alpha_0), 0), \quad \vartheta_i(\alpha_0) = H_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0),$$

$$F(t, z, \mu, \alpha_0) = \frac{1}{\mu^2}\{f(t, \varphi(t, \alpha_0) + \mu z) - f(t, \varphi(t, \alpha_0)) - \mu A(t, \alpha_0) z$$

$$+ \mu g(t, \varphi(t, \alpha_0) + \mu z, \mu) - \mu g(t, \varphi(t, \alpha_0), 0)\},$$

$$V_i(z, \mu, \alpha_0) = \frac{1}{\mu^2}\{Q_i(\varphi(\theta_i, \alpha_0) + \mu z, \alpha_0) - Q_i(\varphi(\theta_i, \alpha_0), \alpha_0) - \mu P_i(\alpha_0) z$$

$$+ \mu H_i(\varphi(\theta_i, \alpha_0) + \mu z, \mu, \alpha_0) - \mu H_i(\varphi(\theta_i, \alpha_0), 0, \alpha_0\}.$$

Denote

$$l(\gamma) = \int_0^{\omega} \Psi(t,\alpha_0)F\big(t,\delta(t,\gamma),0,\alpha_0\big)dt + \sum_{i=1}^{p}\Psi(\theta_i,\alpha_0)V_i\big(\delta(\theta_i,\gamma),0,\alpha_0\big).$$

By the theorem on the existence of periodic solutions of quasilinear systems [8], the necessary and sufficient condition for the existence of an ω-periodic solution to system (10), which tends to the solution $\delta(t,\gamma_0)$ of Eq. (8) as $\mu \to 0$, is the relation $\det(\partial l(\gamma_0)/\partial l\gamma) \neq 0$, where $\gamma = \gamma_0$ is the solution of the equation $l(\gamma) = 0$.

Let us show that the last equation is linear.

First, we note that

$$F_j(t,\delta(t,\gamma),0,\alpha_0) = \frac{1}{2}\sum_{k,r=1}^{n}\frac{\partial^2 f_j}{\partial x_k \partial x_r}\delta_k\delta_r + \sum_{k=1}^{n}\frac{\partial g_j}{\partial x_k}\delta_k + \frac{\partial g_j}{\partial \mu};$$

$$V_{ij}(\delta(\theta_i,\gamma),0,\alpha_0) = \frac{1}{2}\sum_{k,r=1}^{n}\frac{\partial^2 Q_{ij}}{\partial x_k \partial x_r}\delta_k\delta_r + \sum_{k=1}^{n}\frac{\partial H_{ij}}{\partial x_k}\delta_k + \frac{\partial H_{ij}}{\partial \mu}.$$

This implies that

$$l_s(\gamma) = \frac{1}{2}\gamma_q\gamma_\upsilon\Big[\int_0^{\omega}\psi_{js}\frac{\partial^2 f_j}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_r}{\partial\alpha_\upsilon}\,dt + \sum_{i=1}^{p}\psi_{js}\frac{\partial^2 Q_{ji}}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_r}{\partial\alpha_\upsilon}\Big]$$

$$+\gamma_q\Big[\int_0^{\omega}\psi_{js}\Big(\frac{\partial^2 f_i}{\partial x_k\partial x_r}\zeta_r + \frac{\partial g_i}{\partial x_k}\Big)\frac{\partial\varphi_k}{\partial\alpha_q} + \sum_{i=1}^{p}\psi_{js}\Big(\frac{\partial^2 Q_{ji}}{\partial x_k\partial x_r}\zeta_r + \frac{\partial H_{ji}}{\partial x_k}\Big)\frac{\partial\varphi_k}{\partial\alpha_q}\Big] + \ldots, \tag{11}$$

where $s = \overline{1,\ m}$, and the omitted terms do not depend on γ_q.

By differentiating the expression

$$\varphi(t,\alpha) = \varphi(0,\alpha) + \int_0^{t} f(t,\varphi(t,\alpha))dt + \sum_{0<\theta_i<t} I_i(\varphi(\theta_i,\alpha))$$

twice, we can find that the function $\partial^2\varphi/\partial\alpha_q\partial\alpha_r$ for $\alpha = \alpha_0$ is an ω-periodic solution of the equation

$$\begin{aligned}\frac{du}{dt} &= A(t,\alpha_0)u + \sum_{k,r=1}^{n}\frac{\partial^2 f}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_k}{\partial\alpha_\upsilon}, \quad t \neq \theta_i, \\ \Delta u\big|_{t=\theta_i} &= P_i(\alpha_0)u + \sum_{k,r=1}^{n}\frac{\partial^2 Q_i}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_r}{\partial\alpha_\upsilon}.\end{aligned} \tag{12}$$

However, it is known [3] that for system (12) to have an ω-periodic solution, it is necessary and sufficient that the following condition be satisfied:

$$\int_0^{\omega}\Psi(t,\alpha_0)\sum_{k,r=1}^{n}\frac{\partial^2 f}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_r}{\partial\alpha_\upsilon}dt + \sum_{i=1}^{n}\Psi(\theta_i,\alpha_0)\frac{\partial^2 Q_i}{\partial x_k\partial x_r}\frac{\partial\varphi_k}{\partial\alpha_q}\frac{\partial\varphi_r}{\partial\alpha_\upsilon} = 0.$$

This implies that the coefficient by the term $\gamma_q\gamma_r$ is equal to zero. Thus, the vector γ is a solution of the linear system, and the nondegeneracy of its matrix of coefficients is guaranteed by the uniqueness of the ω-periodic solution to system (6) proved above. This completes the proof.

2. Equations with Discontinuous Right-Hand Sides

On the set Ω which was defined in Section 1, we consider a system of differential equations of the type

$$\frac{dx}{dt} = f(t,x) + \mu g(t,x,\mu). \tag{13}$$

Let S_i be the sequence of surfaces determined by the equations $t = t_i(x) + \mu\tau_i(x,\mu)$, in which the functions t_i and τ_i are defined for $x \in \Omega_x$ and $\mu \in (-\mu_0; \mu_0)$; $t_{i+1}(x) - t_i(x) \geq \gamma > 0$ for all $x \in \Omega_x$ and $i = 0, \pm 1, \ldots$.

Denote by Ω_i the domain situated between the surfaces S_i and S_{i+1}. Let A_i and A_{i+1} be open subsets of Ω_i for which the surfaces S_i and S_{i+1}, respectively, are parts of their boundaries. We assume that

$$f \in C^{(0,2)}(\Omega_i) \cap C^{(1,2)}(A_i \cup A_{i+1}), \quad g \in C^{(0,1,1)}(\Omega_i) \cap C^{(1,2,2)}(A_i \cup A_{i+1}),$$

for any i. The functions f and g and their derivatives are continuous up to the sets S_i and S_{i+1} on which they have discontinuities of the first kind. Assume also that there exist a real number $\omega > 0$ and an integer $p > 0$ such that the equalities

$$f(t+\omega, x) = f(t,x), \quad g(t+\omega, x, \mu) = g(t,x,\mu), \quad t_{i+p} = t_i + \omega, \quad \tau_{i+p} = \tau_i$$

hold uniformly in the domain Ω.

Suppose that the generating system

$$\frac{dx}{dt} = f(t,x) \tag{14}$$

has an m-parameter $(m < n)$ family of ω-periodic solutions $\varphi(t,\alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_m)$. For a certain value $\alpha = \alpha_0$, there exists a nonzero minor of the matrix $\partial\varphi(0,\alpha_0)/\partial\alpha$. Suppose that the generating solution is such that

$$1 - \frac{\partial t_i(\varphi(\theta_i,\alpha_0))}{\partial x} f^{\pm} \neq 0,$$

where the signs $+$ and $-$ are used to denote the limit values of functions as $t \to \theta_i \pm$, $x = \varphi(\theta_i, \alpha_0)$, $\alpha = \alpha_0$.

For any i, we realize the following construction. Let $x_0(t)$, $x_0(\theta_i) = x$ be a solution of Eq. (13), and let ξ_i be a solution of the equation $\xi = t_i(x_0(\xi)) + \mu\tau_i(x_0(\xi), \mu)$. For definiteness, we assume that $\xi_i \geq \theta_i$. Preserving smoothness, we extend the functions f and g from the domain Ω_i to the set $\Omega_x \times (\theta_i, \theta_{i+1}) \times (-\mu_0, \mu_0)$ under the condition that these functions and their derivatives should be continuous up to planes $t = \theta_i$ and $t = \theta_{i+1}$. Denote the extensions obtained by f_1 and g_1 and construct the system of differential equations

$$\frac{dx}{dt} = f_1(t,x) + \mu g_1(t,x,\mu).$$

Let $x_1(t)$, $x_1(\xi_i) = x_0(\xi_i)$ be a solution of this system. We define a mapping $\mathcal{J}_i : (x, \mu, \alpha) \to x_1(\theta_i)$. By using the Hadamard lemma and the differentiability of the solutions to system (13) with respect to the initial data and parameter, we find that $\mathcal{J}_i(x, \mu, \alpha) = Q_i(x, \alpha) + \mu H_i(x, \mu, \alpha)$.

Let us construct a system of equations with pulse effects at fixed times:

$$\begin{gathered} \frac{dy}{dt} = f_1(t, y) + \mu g_1(t, y, \mu), \quad t \neq \theta_i, \\ \Delta y\,|_{t=\theta_i} = Q_i(y, \alpha) + \mu H_i(y, \mu, \alpha). \end{gathered} \tag{15}$$

Systems (13) and (15) satisfy the condition (A) defined above for pulse systems (1) and (6).

The system of equations in variations for the solution $\varphi(t, \alpha_0)$ has the form

$$\frac{du}{dt} = A(t, \alpha_0)\, u, \quad t \neq \theta_i, \quad \Delta u\big|_{t=\theta_i} = P_i(\alpha_0)\, u, \tag{16}$$

where

$$A(t, \alpha_0) = \frac{\partial f(t, \varphi(t, \alpha_0))}{\partial x}, \quad P_i(\alpha_0) = (f^- - f^+)\frac{\partial t_i/\partial x}{1 - (\partial t_i/\partial x) f^-},$$

$$\det(E + P_i(\alpha_0)) \neq 0, \quad i = 1, p.$$

Let us construct an ω-periodic solution of the equation

$$\frac{dx}{dt} = A(t, \alpha_0)x + g(t, \varphi(t, \alpha_0), 0), \quad t \neq \theta_i,$$

$$\Delta x\big|_{t=\theta_i} = P_i(\alpha_0)x + H_i(\varphi(\theta_i, \alpha_0), 0)$$

of the form

$$\delta(t, \gamma) = \gamma_1 \frac{\partial \varphi}{\partial \alpha_1} + \ldots + \gamma_m \frac{\partial \varphi}{\partial \alpha_m} + \zeta(t),$$

where $\zeta(t)$ is a special ω-periodic solution of this equation.

Assume that ψ_k, $k = \overline{1, m}$, are ω-periodic linearly independent solutions of the equation conjugated to (16). We compose from these solutions a matrix $\Psi(t, \alpha_0)$ of dimensionality $m \times n$. By analogy with Theorem 1, by virtue of the condition (A), we can prove the following statement.

Theorem 2. *Assume that system* (13) *satisfies the conditions given above. Let the root* $\alpha = \alpha_0$ *of the equation* $\sigma(\alpha) = 0$ *with*

$$\sigma(\alpha) = \int_0^{\omega} \Psi(t, \alpha) g(t, \varphi(t, \alpha), 0)\, dt + \sum_{i=1}^{p} \Psi(\theta_i, \alpha)(f^- - f^+)\frac{\tau_i}{1 - (\partial t_i/\partial x) f^-}$$

satisfy the condition

$$\det\left(\frac{\partial\sigma(\alpha_0)}{\partial\alpha}\right) \neq 0.$$

Then system (13), *for sufficiently small* $|\mu|$, *admits an* ω*-periodic solution of the form* $x = \varphi(t, \alpha_0) + \mu\xi$, *where the function* ξ *tends in B-topology to one of the solutions* $\delta(t, \gamma)$, *as* $\mu \to 0$. *The vector* γ *is defined as a solution of the nondegenerate linear system.*

REFERENCES

1. Yu. A. Mitropol'skii, *Method of Averaging in the Nonlinear Mechanics* [in Russian], Naukova Dumka, Kiev (1971).
2. A.M.Samoilenko, "Method of averaging in systems with pulses," *Mat. Fizika,* Issue 9, 101–117 (1971).
3. A.M.Samoilenko and N. A. Perestyuk, *Differential Systems with Pulse Effects* [in Russian], Vyshcha Shkola, Kiev (1987).
4. A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides* [in Russian], Nauka, Moscow (1985).
5. Yu. I. Neimark and L. P. Shil'nikov, "On application of a small parameter to systems of differential equations with discontinuous right-hand sides," *Izv. Akad. Nauk SSSR. Mekh. Mashinostr.,* No. 6, 51–59 (1959).
6. M. Z. Kozlovskii, "On conditions of the existence of periodic solutions to systems of differential equations with discontinuous right-hand sides including a small parameter," *Prikl. Mat. Mekh.,* **24**, Issue 9, 738–745 (1960).
7. I. G. Malkin, *Some Problems in the Theory of Nonlinear Oscillations* [in Russian], Gostekhizdat, Moscow (1956).
8. A. M. Samoilenko, N. A. Perestyuk and M. U. Akhmetov, *Differential Properties of Solutions and Integral Surfaces of Nonlinear Systems with Pulses* [in Russian], Preprint 90. 37, Institute of Mathematics, Ukrainian Acad. Sci., Kiev (1990).
9. M. U. Akhmetov and N. A. Perestyuk, "On the differential dependence of solutions of systems with pulses of initial conditions," *Ukr. Mat. Zh.,* **41**, No. 8, 1028–1033 (1989).
10. M. U. Akhmetov and N. A. Perestyuk, "On the comparison method for differential systems with pulse effect," *Diff. Uravnen.,* **26**, No. 9, 1475–1483 (1990).