STABILITY OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH IMPULSE ACTION ON SURFACES

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UDC 517.911

We will consider a system of differential equations with the impulse action

$$dx/dt = f(t, x), \quad t \neq t_i(x), \quad \Delta x|_{t=t_i(x)} = I_i(x), \tag{1}$$

in which $x \in \mathbb{R}^n$, and the functions $f(t, x)$, $I_i(x)$, $t_i(x)$ are defined and continuous in t and x, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $i \in \mathbb{Z}$. The function $f(t, x)$ is T-periodic in t, and there exists a number $p \in \mathbb{N}$, for which, for all $x \in \mathbb{R}^n$, $i \in \mathbb{Z}$ we have $I_{i+p}(x) = I_i(x)$, $t_{i+p}(x) = t_i(x) + T$.

The surfaces of the discontinuity satisfy a uniqueness condition which consists in the fact that a solution of Eqs. (1) intersects each of these surfaces no more than once, and also $t_i(x) > t_{i-1}(x)$, $x \in \mathbb{R}^n$, $i \in \mathbb{Z}$.

The question of the asymptotic stability of the family of periodic solutions of (1) depending on $\gamma \in \mathbb{R}^m$, is analyzed here. Beforehand, a theorem was proved on the differentiability with respect to the parameter of the solutions of the system

$$\begin{gathered} dx/dt = f(t, x, \mu), \quad t \neq t_i(x, \mu), \\ \Delta x|_{t=t_i(x, \mu)} = I_i(x, \mu), \end{gathered} \tag{2}$$

where the functions $f(t, x, \mu)$, $I_i(x, \mu)$, and $t_i(x, \mu)$ are defined and continuous for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, $i \in \mathbb{Z}$ with respect to t, x, and μ; the surface $t = t_i(x, \mu)$ satisfy a uniqueness condition and $t_i(x, \mu) > t_{i-1}(x, \mu)$ for $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, $i \in \mathbb{Z}$.

Differential equations with impulse action arose as mathematical models in the solution of practical problems of importance and now the theory of impulse systems finds wide application in nonlinear mechanics [1-3]. In this paper, the results obtained are used to prove the asymptotic stability of periodic motion of the vibroimpact system described in [4].

Let $x(t)$ be a solution of (1) [or (2)], defined on the interval I (I can be the real semiaxis or line).

We will say that a solution $y(t)$ of this equation lies in an ε-neighborhood of a solution $x(t)$ if: 1) the measure of the symmetric difference of the domains of existence of x and y is not larger than ε; 2) the points of discontinuity of y are located in an ε-neighborhood of the points of discontinuity of x; 3) for all t lying outside ε-neighborhoods of the points of discontinuity of $x(t)$, we have $\|y(t) - x(t)\| < \varepsilon$.

We will call the topology defined using ε-neighborhoods a B-topology.

We will call a solution $x(t)$ B-stable if it is defined for $t \geq t_0$, and for any $\varepsilon > 0$ there is $\delta > 0$ such that a solution $y(t)$ satisfying $\|y(t_0) - x(t_0)\| < \varepsilon$ belongs to an ε-neighborood of $x(t)$.

A B-stable solution is B-asymptotically stable if there exists $\delta > 0$ such that for any $\varepsilon > 0$ there is a number $\theta > t_0$ such that a solution y for which $\|y(t_0) - x(t_0)\| < \delta$, belongs to an ε-neighborhood of $x(t)$ for $t \geq \theta$.

One can verify that a solution of (2) depends continuously in the B-topology on the initial data t_0 and x_0 and the parameter μ in any compactum from $\mathbb{R}^{1+n+m}$, containing (t_0, x_0, μ) only if $t_0 \neq t_1(x_0, \mu)$ for all $i \in \mathbb{Z}$.

Let us now assume that in the domain

$$F = \{(x, t, \mu) \mid \|x - x_0\| < d, \quad t_0 < t < t_0 + T, \quad \|\mu - \mu_0\| < b\}$$

University of Kiev. Aktiubinsk Pedagogical Institute. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 12, pp.1596-1601, December, 1989. Original article submitted February 4, 1988.

the functions $f(t, x, \mu)$, $I_i(x, \mu)$, $t_i(x, \mu)$ have continuous and bounded derivatives with respect to x_j and μ_k, $j = \overline{1, n}$, $k = \overline{1, m}$

Let the solution $x(t) = x(t, \mu_0)$, $x(t_0) = x_0$ of (2) be defined for all $t \in (t_0, t_0 + T)$. The solution $y(t) = x(t, \mu_0 + \Delta\mu_1)$, $\Delta\mu_1 = (\Delta\mu, 0, \ldots, 0)$ also satisfies the initial condition $y(t_0) = x_0$.

Let us call a piecewise-continuous function $u(t)$ a B-derivative of $x(t)$ with respect to μ_1 if $y(t)$ is located in a θ-neighborhood of $x(t) + \Delta\mu u(t)$, $\theta = O(\Delta\mu)$ and, furthermore, for all t, located outside θ-neighborhoods of the points of discontinuity of $x(t)$, we have $\|y(t) - x(t) - \Delta\mu u(t)\| < \nu$, $\nu = o(\Delta\mu)$.

The B-derivatives with respect to all the remaining μ_k, $k = \overline{2, m}$, are similarly defined.

Let τ_i, $i = \overline{1, p}$ be the points of discontinuity of $x(t)$ belonging to $(t_0, t_0 + T)$. Let us denote

$$A(t) = \frac{\partial f(t, x(t), \mu_0)}{\partial x}, \quad P_i = V_i - W_i + \frac{\partial I_i(x(\tau_i), \mu_0)}{\partial x}(E + V_i),$$

$$g_j(t) = \frac{\partial f(t, x(t), \mu_0)}{\partial \mu_j}, \tag{3}$$

$$J_i^j = \frac{\dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial \mu_j}(f(\tau_i, x(\tau_i), \mu_0) - f(\tau_i, x(\tau_i +), \mu_0))}{1 - \langle \dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, f(\tau_i, x(\tau_i), \mu_0)\rangle} + \frac{\partial I_i(x(\tau_i), \mu_0)}{\partial \mu_j}.$$

The matrices V_i and W_i are such that for any $z \in \mathbb{R}^n$

$$V_i z = \frac{\langle \dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, z\rangle f(\tau_i, x(\tau_i), \mu_0)}{1 - \langle \dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, f(\tau_i, x(\tau_i), \mu_0)\rangle},$$

$$W_i z = \frac{\dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, z\rangle f(\tau_i, x(\tau_i +), \mu_0)}{1 - \langle \dfrac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, f(\tau_i, x(\tau_i), \mu_0)\rangle}.$$

The scalar product in $\langle\ ,\ \rangle$ is denoted by the angular brackets

THEOREM 1. If system (2) satisfies all the conditions enumerated above, then for each μ_j, $j = \overline{1, m}$, there exists a B-derivative of $x(t)$ which is a solution of a linear system of differential equations with the impulse action

$$\frac{du}{dt} = A(t)u + g_j(t), \quad t \neq \tau_i, \quad \Delta u|_{t=\tau_i} = P_i u + J_i^j \tag{4}$$

with initial condition $u(t_0) = 0$.

Proof. Let us verify the theorem for $j = 1$. Let θ_i be the points of discontinuity of $y(t)$. For simplicity, without destroying generality, we will let $\theta_i \geq \tau_i$ for all i.

Let us denote $\nu(t) = y(t) - x(t) - \Delta\mu u(t)$, where $u(t)$ is a solution of (4) for $j = 1$ with $u(t_0) = 0$.

The solutions $x(t)$, $y(t)$, $u(t)$ have the integral representations

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau), \mu_0)\, d\tau + \sum_{t_0 < \tau_i < t} I_i(x(\tau_i), \mu_0),$$

$$y(t) = x_0 + \int_{t_0}^{t} f(\tau, y(\tau), \mu_0)\, d\tau + \sum_{t_0 < \theta_i < t} I_i(y(\theta_i), \mu),$$

$$u(t) = \int_{t_0}^{t} (A(\tau) u(\tau) + g_1(\tau))\, d\tau + \sum_{t_0 < \tau_i < t} (P_i u(\tau_i) + J_i^1). \tag{5}$$

Using (5) and the differentiability of $f(t, x, \mu)$, $I_i(x, \mu)$, and $t_i(x, \mu)$, we find that for all $t \bar{\in} (\tau_i, \theta_i)$

$$\| v(t) \| \leqslant e^{kT} (p w(\Delta\mu) + \Delta\mu \xi(\Delta\mu) T), \tag{6}$$

where $k = \sup_{t_0 \leqslant t \leqslant t_0 + T} \| A(t) \|$, $w = o(\Delta\mu)$, $\xi(\Delta\mu) = o(\Delta\mu)$.

Furthermore, for all $\Delta t_i = \theta_i - \tau_i$, $i = \overline{1, p}$, we have

$$\Delta t_i = \frac{\Delta\mu \left(\langle \frac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, u(\tau_i) \rangle + \frac{\partial t_i(x(\tau_i), \mu_0)}{\partial \mu_1} \right)}{1 - \langle \frac{\partial t_i(x(\tau_i), \mu_0)}{\partial x}, f(\tau_i, x(\tau_i), \mu_0) \rangle} + o(\Delta\mu). \tag{7}$$

Relations (6) and (7) prove the theorem for $j = 1$. Its validity is verified in the same way for all remaining $j = \overline{2, m}$. The theorem is proved.

Now let us assume that in (1) the functions $f(t, x)$, $I_1(x)$, and $t_i(x)$ are continuously differentiable in x_k, $k = \overline{1, n}$.

Let Eq. (1) have a family of T-periodic solutions $x(t, \gamma)$ depending on γ in some compactum $K \subset \mathbb{R}^m$, $m < n$, inside K for each γ_j, $j = 1, m$, and B-derivatives $\varphi_j(t) = \partial x(t, \gamma)/\partial \varepsilon_j$. Let us assume that these derivatives are linearly independent for $t = 0$. Without destroying generality, we can consider that each point $x(0, \gamma)$ does not belong to one of the surfaces of discontinuity, along with some neighborhood of it.

For an arbitrary solution $x(t)$ of (1) let us define $z(t) = x(t) - x(t, \gamma)$.

Let us denote by $\tau_i = \tau_i(\gamma)$ the points of discontinuity of $x(t, \gamma)$, and by θ_i the points of discontinuity of $x(t)$. Let $\theta_i \geq \tau_i$, $i = \overline{1, p}$.

Applying the method of proving Theorem 1, one can show that for $t \bar{\in} (\tau_i, \theta_i)$ the function $z(t)$ coincides with the solution of

$$du/dt = A(t) u + g(t, u), \quad t \neq \tau_i, \quad \Delta u |_{t=\tau_i} = P_i u + J_i(u), \tag{8}$$

in which the matrices $A(t)$ and P_i and the functions $g(t, u)$ and $J_i(u)$ depend on γ. The matrices $A(t)$ and P_i are determined by formulas (3), where $x(t) = x(t, \gamma)$. The conditions $\|g(t, u)\| = o(\|u\|)$ and $\|J_i(u)\| = o(\|u\|)$ are satisfied,

The next proposition is the analog of a theorem from [6].

<u>THEOREM 2.</u> For each $\gamma \in K$ among the characteristic exponents of the system

$$dv/dt = A(t) v, \quad t \neq \tau_i, \quad \Delta v |_{t=\tau_i} = P_i v \tag{9}$$

let there be $n - m$, having negative real parts.

Then there exists a real manifold $W(\gamma)$ of dimension $n - m$, containing a point $x(0, \gamma)$ such that a T-periodic solution $x(t, \gamma)$ is conditionally B-asymptotically stable with respect to $W(\gamma)$.

<u>Proof.</u> By Theorem 1, the B-derivatives φ_j are T-periodic solutions of (9). Since the functions φ_j are linearly independent for $t = 0$, they are linearly independent solutions of this system [5]. Therefore, Eqs. (9) have a characteristic exponent equal to zero, with a multiplicity not equal precisely to m.

Let us denote by $Y(t)$ the (normed at zero) fundamental matrix of the solutions of system (9), and C the matrix for which

$$Y(t + T) = Y(t) C. \tag{10}$$

Without destroying the generality of subsequent arguments, one can consider that

$$C = e^{BT}, \tag{11}$$

where $B = \mathrm{diag}[0, B_1]$, B_1 is a matrix of dimension $(n - m) \times (n - m)$, all the eigenvalues of which have negative real parts. Let us denote $Z(t) = Y(t)e^{-Bt}$, $V_1(t, s) = Z(t)\,\mathrm{diag}\cdot [0, e^{B_1(t-s)}]Z^{-1}(s)$, $V_2(t, s) = Z(t)\,\mathrm{diag}[E, 0,]Z^{-1}(s)$. It follows from the expressions for V_1 and V_2 that there exists constants $M > 1$ and $\alpha < 0$, for which for $t > S$

$$\|V_1(t, s)\| \leqslant Me^{\alpha(t-s)}, \quad \|V_2(t, s)\| \leqslant M. \tag{12}$$

One can verify that the solution of the integral equation

$$\zeta(t) = Y(t)\,a + \int_0^t V_1(t, s)\,g(\zeta, s)\,ds - \int_t^\infty V_2(t, s)\,g(\zeta, s)\,ds + \sum_{0<\tau_i<t} V_1(t, \tau_i)\,J_i(\zeta(\tau_i)) - \sum_{t<\tau_i} V_2(t, \tau_i)\,J_i(\zeta(\tau_i)) \tag{13}$$

is also a solution of (8).

With the help of (12) and the properties of $g(t, x)$ and $J_i(x)$ it is proved by the method of sequential approximations that for sufficiently small h, $\|a\| < h$, there exists a solution $\zeta(t)$ of (13) for which

$$\|\zeta(t)\| < M_1 \exp(\alpha t/2)$$

uniformly with respect to t and a, $t > 0$, $\|a\| < h$, $0 < M_1 < +\infty$.

Substituting in (13) $t = 0$, we obtain that $\zeta_i(0) = a_i$ if $i > m$, and $\zeta_i = \Phi_i(a_{m+1}, a_{m+2}, \ldots, a_n)$, $i = \overline{1, m}$, where $\Phi_i(a_{m+1}, \ldots, a_n) = o(\|a\|)$. This means

$$\zeta_i = \Phi_i(\zeta_{m+1}, \zeta_{m+2}, \ldots, \zeta_n) \tag{14}$$

and

$$\Phi_i(\zeta_{m+1}, \zeta_{m+2}, \ldots, \zeta_n) = o(\|\zeta\|). \tag{15}$$

The relations (14) determine a manifold $W(\gamma)$, each point $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ of which, situated in a sufficiently small neighborhood of the origin of the coordinates, is an initial value of a solution of (8), approaching zero as $t \to +\infty$. Hence, the proof of the assertion of the theorem follows from the continuous dependence in the B-topology of the solutions of (1) on the initial data.

A corollary of Theorem 2 is an analog of the Lyapunov–Poincaré theorem on the asymptotic stability of a unique T-periodic solution of (1).

Let us consider the dynamic system analyzed in [4]. It consists of a globule bouncing on a platform moving vertically. It is assumed that the platform does not respond to the impacts of the globule and moves according to the law $X = X_0 \sin \omega t$.

The motion of the globule between kicks is determined by the formula $x = 1/2gt^2 + \dot{x}^0 t + x^0$. It was proved in [4] that under the condition

$$\omega^2 > \frac{\pi g}{X_0}\,\frac{1 - R}{1 + R}, \tag{16}$$

where R is the restoration coefficient, such a dynamic model admits two periodic motions $x_1(t)$ and $x_2(t)$ with a period equal to the period of oscillations of the platform.

It is assumed that the periodic motions experience an impulse action once in the period, at the moments $t = \varphi_1$ and $t = \varphi_2$, $0 < \varphi_1, \varphi_2 < 2\pi/\omega$, respectively.

Using the results of [4], we find that the mathematical model for analyzing a system on $[0, 2\pi/\omega]$ has the form of differential equations with the impulse action

$$\begin{gathered} dx_1/dt = x_2, \quad dx_2/dt = -g, \quad t \neq \frac{1}{\omega}\arcsin\frac{x_1}{X_0} \equiv t_0(x), \quad \Delta x_1|_{t=t_0(x_1)} = 0, \\ \Delta x_2|_{t=t_0(x_1)} = (1 + R)\left[X_0\omega\cos\left(\arcsin\frac{x_1}{X_0} - \dot{x}^0 + \frac{g}{\omega}\arcsin\frac{x_1}{X_0}\right],\right. \end{gathered} \tag{17}$$

where $\dot{x}^0$ is the velocity at $t = 0$.

The systems of linear equations

$$du_1/dt = u_2, \quad du_2/dt = -g, \quad t \neq \varphi_1, \quad \Delta u_1|_{t=\varphi_1} = 0, \quad \Delta u_2|_{t=\varphi_1} = b_+ u_1, \tag{18}$$

and

$$du_1/dt = u_2, \quad du_2/dt = -g, \quad t \neq \varphi_2, \quad \Delta u_1|_{t=\varphi_2} = 0, \quad \Delta u_2|_{t=\varphi_2} = b_- u_1, \tag{19}$$

where

$$b_+ = \frac{(1+R)^2\omega}{\pi(1-R)} - (1+R)\omega\sqrt{\left[\frac{X_0\omega^2(1+R)}{\pi g(1-R)}\right]^2 - 1},$$

$$b_- = \frac{(1+R)^2\omega}{\pi(1-R)} + (1+R)\omega\sqrt{\left[\frac{X_0\omega^2(1+R)}{\pi g(1-R)}\right]^2 - 1}$$

are, respectively, systems of equations in variations for equations (17) with respect to $x_1(t)$ and $x_2(t)$.

The characteristic equation [5] for Eqs. (18) and (19), respectively, has the form

$$\lambda^2 - 2\left(1 + \frac{\pi b_+}{\omega}\right)\lambda + 1 = 0 \tag{20}$$

and

$$\lambda^2 - 2\left(1 + \frac{\pi b_-}{\omega}\right)\lambda + 1 = 0. \tag{21}$$

Solving these equations, we obtain that Eq. (21) does not have roots with a nonpositive real part. Consequently, the solution $x_2(t)$ is unstable.

Both roots of (20) have negative real parts if

$$\omega^2 > \sqrt{\left[\frac{\pi g(1-R)}{X_0(1+R)}\right]^2 + \left[\frac{g}{X_0}\left(1 + \frac{1-R}{(1+R)^2}\right)\right]^2}. \tag{22}$$

Comparing (16) and (22), we find that by the analog of the Lyapunov–Poincaré theorem on the stability of a periodic solution, if (22) is valid, the vibroimpact system under investigation admits a unique B-asymptotically stable $2\pi/\omega$-periodic solution.

The result obtained agrees well with the deduction of [4].

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