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DIFFERENTIABLE DEPENDENCE OF THE SOLUTIONS OF IMPULSE SYSTEMS ON INITIAL DATA

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In several applied problems [1-3] one encounters systems undergoing an impulse action at the moment when the solution attains definite points in the extended phase space. Meanwhile, such systems, unlike the equations in which the solutions are subjected to impulses at fixed instants of time, selected in a random manner [4, 5], have been inadequately investigated.

In this paper we investigate the fundamental properties of differential equations with impulse action on surfaces: existence and uniqueness of solutions, dependence of solutions on the initial data and on parameters. Afterward, the obtained results are applied to the study of periodic impulse systems.

We consider the system of differential equations with impulse action

$$dx/dt = f(t, x), \quad t \neq t_i(x), \quad \Delta x|_{t=t_i(x)} = I_i(x), \tag{1}$$

in which $x \in R^n$, the functions $f(t, x)$, $I_i(x)$, $t_i(x)$ are defined for all $t \in R$, $x \in R^n$, $i \in Z$ (Z is the set of integers) and are continuous with respect to t and x. Each compactum from $R \times R^n$ intersects a finite number of surfaces $t = t_i(x)$, and $t_i(x) > t_{i-1}(x)$ for all $x \in R^n$ and $i \in Z$. Let $(t_0, x_0) \in R \times R^n$. We separate a bounded closed set $F\{(x, t, i) \mid \|x - x_0\| \leq d,\ t_0 \leq t \leq t_0 + T,\ i = \overline{i_0, i_{0+p}}\}$, where $t = t_i(x)$, $i = \overline{i_0, i_0 + p}$, are all the surfaces intersecting with the domain (t, x): $t_0 \leq t \leq t_0 + T$, $\|x - x_0\| \leq d$.

Let $M = \max\{\max_F \| f(t, x) \|, \max_F \| I_i(x) \|\}$. We also assume that each solution of the system (1) intersects any of the surfaces $t = t_i(x)$ at most once. We denote by $i(t_0, \zeta)$ the number of surfaces $t = t_i(x)$ having at least one common point with the domain (t, x): $t_0 \leq t \leq t_0 + \tau$, $\|x - x_0\| \leq d$. Let $h = \min(T, t')$, where t' is the supremum of the set of the solutions of the inequality $t + i(t_0, t) \leq d/M$.

Making use of the property of the mappings $I_i(x)$ and the theorem of the existence and uniqueness of the solution for ordinary differential equations, one can show the validity of the following theorem.

THEOREM 1. If the functions $f(t, x)$ and $I_i(x)$ satisfy uniformly on the set F a Lipschitz condition, then the Cauchy problem $x(t_0) = x_0$ for the system (1) on the interval $[t_0, t_0 + h]$ has a unique solution.

Let $x_j(t)$, $j = 1, 2$, be solutions of the system (1), let t_i^j be the points of discontinuity of these solutions, i.e., the solutions of the equations $t = t_i(x_j(t))$, $i = \overline{i_0, i_0 + p}$. Since, in general, the points t_i^1 and t_i^2 do not coincide, we cannot talk about the uniform nearness of these solutions with respect to all t. Therefore, for the piecewise-continuous functions, considered in this paper, we define the following topology.

We shall say that a solution $x_1(t)$ is in the ε-neighborhood of a solution $x_2(t)$ if: 1) the measure of the symmetric difference of the domains of existence of these solutions

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does not exceed ε; 2) for all i we have the inequality $|t_i^1 - t_i^2| < \varepsilon$; 3) the inequality $\|x_1(t) - x_2(t)\| < \varepsilon$ is valid for all t satisfying the condition $|t - t_i^2| > \varepsilon$.

The topology defined with the aid of ε-neighborhoods will be called the B-topology. It is Hausdorff and it can be constructed also in the case when the solutions x_1 and x_2 are defined on the semiaxis or on the entire real axis.

Topologies for piecewise-continuous functions have been defined for the first time in [6, 7]. In [8], the concept of the ε-neighborhood is applied implicitly to the definition of a discontinuous almost-periodic function.

THEOREM 2. If the differential equation with impulse action

$$dx/dt = f(t, x, \mu), t \neq t_i(x, \mu), \Delta x|_{t=t_i(x,\mu)} = I_i(x, \mu) \tag{2}$$

satisfies, independent of the parameter $\mu \in R^m$, all the conditions indicated for the system (1) and, in addition, depends continuously on this parameter, then the solution $x(t, \mu)$, with the initial condition $x(t_0, \mu) = x_0$, depends continuously in the B-topology on μ at each point μ_0 such that $t_0 \neq t_i(x_0, \mu_0)$.

Remark. The point (t_0, x_0) should not belong to any of the surfaces $t = t_i(x, \mu_0)$. Otherwise, one can select a value μ, arbitrarily close to μ_0, for which $t_k(x_0, \mu) < t_0$ under the simultaneous condition $t_0 = t_k(x_0, \mu_0)$; thus, already at the moment t_k + the solution $x(t, \mu)$ differs from the solution $x(t, \mu_0)$ by a quantity of order $I_k(x_0, \mu_0)$.

The proof of Theorem 2 is similar to the proof of the theorem on the continuous dependence for ordinary differential equations.

The proof of the continuous dependence of the system (1) on the initial data t_0 and x_0 reduces to the investigation of the dependence on a parameter.

We continue the investigation of the system (1), assuming, in addition, that inside the domain F there exist the continuous derivatives $\partial f(t, x)/\partial x_j$, $\partial I_i(x)/\partial x_j$, $\partial t_i(x)/\partial x_j$, $j = \overline{1, n}$.

We shall say that a piecewise continuous function $u_j(t)$ is the B-derivative of the solution $x(t, t_0, x_0)$ of the system (1) with respect to x_0^j, $x_0 = (x_0^1, \ldots, x_0^j, \ldots, x_0^n)$, if the function $\xi u_j(t)$ is in the θ-neighborhood of the difference $x_j(t) - x(t)$, where $x_j(t)$ is the solution of Eq. (1) with the initial condition $x_j(t_0) = (x_0^1, \ldots, x_0^j + \xi, \ldots, x_0^n)$ and $\theta \to 0$ for $\xi \to 0$. In addition, for all t from $[t_0, t_0 + T]$, lying outside the θ-neighborhoods of the points of discontinuity of the solution $x(t)$, we have the inequality $\|x_j - x - \xi u_j\| < \theta_1$, where θ_1 is an infinitely small quantity of higher order than ξ.

In a similar manner one defines the B-derivatives with respect to t_0 also in the case of system (2) relative to the parameters μ_k, $k = \overline{1, m}$.

THEOREM 3. The solution $x(t)$ of system (1), satisfying all the above-mentioned conditions, has B-derivatives with respect to t_0 and x_0^i, which are solutions of the system

$$du/dt = A(t)u, t \neq \tau_i, \quad \Delta u|_{t=\tau_i} = P_i u, \tag{3}$$

with the initial values $f(t_0, x_0)$ and $e^i = (\underbrace{0, \ldots 0, 1}_{i}, 0, \ldots, 0)$.

In system (3), τ_i are the points of discontinuity of the solution $x(t)$, $A(t) = \partial f(t, x(t))/\partial x$, $P_i = V_i - W_i + [\partial I_i(x(\tau_i))/\partial x](E + V_i)$, where V_i and W_i are matrices such that for each $z \in R^n$ we have the equalities

$$V_i z = \frac{\langle \partial t_i(x(\tau_i))/\partial x, z\rangle}{1 - \langle \partial t_i(x(\tau_i))/\partial x, f(\tau_i\, x(\tau_i))\rangle} f(\tau_i, x(\tau_i)),$$

$$W_i z = \frac{\langle \frac{\partial t_i(x(\tau_i))}{\partial x}, z\rangle}{1 - \langle \frac{\partial t_i(x(\tau_i))}{\partial x}, f(\tau_i, x(\tau_i))\rangle} f(\tau_i, x(\tau_i +)).$$

Proof. First we prove the theorem for x_0^1. According to Theorem 2, there exists an infinitely small quantity $V(\xi)$ such that the solution $x_1(t) = x(t, t_0, x_0 + \xi)$, $\xi = (\xi, 0, \ldots, 0)$ is in the $V(\xi)$-neighborhood of the solution $x(t)$, defined on the segment $[t_0, t_0 + T]$.

Let θ_i be points of discontinuity of the solution $x_1(t)$. For the sake of simplicity, without loss of generality, we shall assume that $\theta_i \geq \tau_i$. By the assumptions of the theorem, for the points $t \in U(\tau_i, \theta_i)$ we have

$$f(t, x_1(t)) - f(t, x(t)) = (A(t) + Q(\xi))(x_1(t) - x(t)),$$
$$I_i(x_1(\tau_i)) - I_i(x(\tau_i)) = (\partial I_i(x(\tau_i))/\partial x + R(\xi))(x_1(\tau_i) - x(\tau_i)),$$
$$t_i(x_1(\theta_i)) - t_i(x(\tau_i)) = \langle \partial t_i(x(\tau_i))/\partial x + r(\xi), x_1(\theta_i) - x(\tau_i)\rangle,$$

where $\|Q(\xi)\| < \alpha$, $\|R(\xi)\| < \alpha$, $\|r(\xi)\| < \alpha$, α being an infinitely small quantity.

The solutions $x(t)$ and $x_1(t)$ have the integral representations

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\,d\tau + \sum_{t_0 \leqslant \tau_i < t} I_i(x(\tau_i)),$$

$$x_1(t) = x_0 + \xi + \int_{t_0}^{t} f(\tau, x_1(\tau))\,d\tau + \sum_{t_0 \leqslant \theta_i < t} I_i(x_1(\theta_i)).$$

Let $u_1(t)$ be a solution of the equations (3) with the initial value $u_1(t_0) = e^1$. We denote $k = \sup_{t_0 \leqslant t \leqslant t_0+T} \|A(t)\|$, $\nu(t) = x_1(t) - x(t) - \xi u_1(t)$.

If $t \in [t_0, \tau_1]$, then we find $\|\nu(t)\| \leqslant \alpha V(\xi)(\tau_1 - t_0) + \int_{t_0}^{t} k\|\nu(\tau)\|\,d\tau$. From here, by the Gronwall–Bellman lemma, there follows that

$$\|\nu(t)\| \leqslant \alpha V(\xi)(\tau_1 - t_0)\, e^{k(\tau_1 - t_0)}. \tag{4}$$

Let $\Delta t = \theta_1 - \tau_1$. Making use of the differentiability of the discontinuity surfaces and of inequality (4), we obtain

$$\Delta t = \langle \partial t_1(x(\tau_1))/\partial x + r(\xi),\ \ \xi u_1(\tau_1) + \Delta t\,(f(\tau_1, x(\tau_1)) + \xi A(\tau_1)\,u(\tau_1) + p_0(\xi))\rangle,$$

where p_0 is an infinitely small quantity.

From here it follows that there exists an infinitely small quantity $p_1(\xi)$, of higher order than ξ, for which we have the equality

$$\Delta t = \frac{\xi\,\langle \partial t_1(x(\tau_1))/\partial x, u_1(\tau_1)\rangle}{1 - \langle \partial t_1(x(\tau_1))/\partial x, f(\tau_1, x(\tau_1))\rangle} + p_1(\xi). \tag{5}$$

Now we estimate the difference $\nu(\theta_1+) - \nu(\tau_1)$. We have

$$\nu(\theta_1+) - \nu(\tau_1) = \int_{\tau}^{\theta_1} (f(\tau, x_1(\tau)) - f(\tau, x(\tau)) - \xi A(\tau)\,u_1(\tau))\,d\tau +$$
$$+ I_1(x_1(\theta_1)) - I_1(x(\tau_1)) - \xi P_1 u_1(\tau_1) = \Delta t\,(f(\tau_1, x(\tau_1) + \xi u(\tau_1) + p_2(\xi)) -$$
$$- f(\tau_1, x(\tau_1) + I_1(x(\tau_1)) + p_3(\xi)) - \xi\,(A(\tau_1)\,u(\tau_1) + p_4(\xi)) +$$
$$+ I_1(x(\tau_1) + \xi u(\tau_1) + \int_{\tau_1}^{\theta_1} f(\tau, x_1(\tau))\,d\tau) - I_1(x(\tau_1)) - \xi P_1 u_1(\tau_1),$$

where p_2, p_3, and p_4 are infinitely small quantities.

From the last equality, by virtue of the boundedness of the function u_1, there follows that $\nu(\theta_1+) - \nu(\tau_1) = W(\xi)$, where $\|W(\xi)\|/\xi \to 0$ for $\xi \to 0$. Thus, $\|\nu(\xi_1+)\| \leq W(\xi) + \alpha V(\xi)\cdot(\tau_1 - t_0)e^{k(\tau_1 - t_0)}$.

In view of the compactness of F, we can assume that $W(\xi)$ does not depend on τ_i. Therefore, applying the above-obtained estimate p times, we find that for any $t \in \bigcup_{i=1}^{p} (\tau_i, \theta_i)$ we have the inequality $\|\nu(t)\| < e^{kT}(pW(\xi) + \alpha V(\xi)T)$, which, together with relation (5), proves the theorem for x_0^1. In the same manner its validity is proved for all the remaining x_0^j, $j = \overline{2, n}$.

Applying B-derivatives with respect to x_0^j, one can show, just as for ordinary differential equations, that the theorem holds for t_0.

Now we make use of the obtained results for the investigation of periodic systems. We consider the quasilinear system of differential equations with impulse action

$$dx/dt = A(t)x + f(t) + \mu\varphi(t, x), t \neq t_i + \mu t_i(x),$$
$$\Delta x|_{t=t_i+\mu t_i(x)} = B_i x + g_i + \mu\psi_i(x), \tag{6}$$

in which A(t) and f(t) are a matrix and a vector function, continuous and periodic with period T, while $\varphi(t, x) \in C^{(0,1)}_{(t,x)}(R \times R^n)$, $\psi_i(x), t_i(x) \in C^{(1)}(R^n)$, $\varphi(t+T, x) = \varphi(t, x)$, $\psi_{i+p}(x) = \psi_i(x)$, $t_{i+p}(x) = t_i(x)$, $t_{i+p} = t_i + T$, $B_{i+p} = B_i$, $g_{i+p} = g_i$, $i \in Z$, and μ is a small parameter.

THEOREM 4. Assume that the system

$$dy/dt = A(t)y + f(t), t \neq t_i, \Delta y|_{t=t_i} = B_i y + g_i \tag{7}$$

has a unique T-periodic solution $y_0(t)$. Then, for sufficiently small μ, system (6) has a unique T-periodic solution, which for $\mu \to 0$ tends to the solution $y_0(t)$ in the B-topology.

Proof. Let $y(t, \eta, \mu)$ be a solution of system (6), satisfying the initial condition $y(0, \eta, \mu) = \eta$, and let $y_0(t) = y(t, \eta_0, 0)$ be the periodic solution, with period T, of the generating system (7). Without loss of generality, we can assume that the point $(0, \eta_0)$, together with some neighborhood of it, does not belong to any of the planes $t = t_i$. In order that for a sufficiently small μ the solution $y(t, \eta, \mu)$ be T-periodic, it is necessary and sufficient that the equation

$$y(T, \eta, \mu) - h = 0 \tag{8}$$

be solvable with respect to η.

Let $D(\eta, \mu) = y(T, \eta, \mu) - \eta$. We show that the determinant $D_\eta'(\eta_0, 0)$ exists and it is different from zero. Indeed, by Theorem 3, there exist the B-derivatives $\partial y(t, \eta, \mu)/\partial\eta_j$, $j = \overline{1, n}$. Let $Z(t, \eta, \mu) = (\partial y_i \partial\eta_k)$, $i, k = \overline{1, n}$.

For a sufficiently small μ, the point $(0, \eta_0)$ is not on any of the surfaces $t = t_i + \mu t_i(x)$ together with some neighborhood of it. In this neighborhood, the B-derivative coincides with the usual derivative.

Differentiating system (6) with respect to η, we can see that $Z(t, \eta_0, 0)$ is a normalized fundamental matrix for (7). On the other hand, $D_\eta^1(\eta_0, 0) = \det(Z(T, \eta_0, 0) - E)$ and, since by virtue of the conditions of the theorem, the eigenvalues of the matrix $Z(T, \eta_0, 0)$ are different from unity [4], it follows that $D_\eta^1(\eta_0, 0) \neq 0$. Therefore, in a sufficiently small neighborhood of the point $(0, \eta_0)$, Eq. (8) is solvable with respect to η. The existence and the uniqueness of a T-periodic solution are proved. The fact that the solution $y(t, \eta, \mu)$ tends to $y_0(t)$ when $\mu \to 0$ follows from Theorem 2. The theorem is proved.

Assume that system (1) satisfies the conditions of Theorem 3, it is periodic with period T, $f(t + T, x) = f(t, x)$, $I_{i+p}(x) = I_i(x)$, $t_{i+p}(x) = t_i(x) + T$. We assume that system (1) has a T-periodic solution $x_0(t)$.

We shall say that a solution $x_0(t)$ is B-stable if it is defined on the segment $[t_0, +\infty)$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution $x_1(t)$, satisfying the condition $\|x_1(t_0) - x_0(t_0)\| < \delta$, will be in the ε-neighborhood of the solution $x_0(t)$.

A solution $x_0(t)$ is said to be B-asymptotically stable if it is B-stable and there exists $\delta > 0$ such that for any $\varepsilon > 0$ one can find a number $\theta > t_0$ such that the solution $x_1(t)$, for which $\|x_1(t_0) - x_0(t)\| < \delta$, is in the ε-neighborhood of the solution x_0, $t > \theta$.

We prove the following statement, which is an analog of the Lyapunov–Poincaré theorem.

THEOREM 5. If all the multipliers of the equation in variations (3) for the solution $x_0(t)$ are less than unity in absolute value, then the solution $x_0(t)$ is asymptotically stable.

Proof. Let $x_1(t)$ be a solution of Eqs. (1) with the initial condition $x_1(t_0) = x_0 + \xi$. We denote by τ_i the points of discontinuity of the solution $x_0(t)$, and by θ_i the points of discontinuity of the solution $x_1(t)$. Assume that $\theta_i \geq \tau_i$ for all i. We denote $\Delta t_i =$

$\theta_i - \tau_i$. Making use of the theorem on the continuous dependence of the solution on the initial data, we find that, for any i such that $\tau_i \in [t_0, t_0 + T]$, we have the equality

$$\Delta t_i = \frac{\langle \partial t_i (x_0(\tau_i))/\partial x, x_1(\tau_i) - x_0(\tau_i)\rangle}{1 - \langle \partial t_i (x_0(\tau_i))/\partial x, f(\tau_i x_0(\tau_i))\rangle} + r_1(\xi), \tag{9}$$

where $r_1(\xi) \to 0$ for $\xi \to 0$. We note that $r_1(\xi)$ does not depend on t_0.

The difference $x_1(t) - x_0(t)$ at the points $t \bar{\in} \cup (\tau_i, \theta_i)$ coincides with the solution of the system

$$\begin{aligned} du/dt &= f(t, x_0(t) + u) - f(t, x_0(t)), t \neq \tau_i, \\ \Delta u|_{t=\tau_i} &= I_i(x_0(\tau_i)) + u + \int_{\tau_i}^{\theta_i} f(\tau, x_0(\tau) + u)\, d\tau) - I_i(x_0(\tau_i)) + \\ &+ \int_{\tau_i}^{\theta_i} f(\tau, x_0(\tau) + u)\, d\tau + \int_{\theta_i}^{\tau_i} f(\tau, x_2(\tau))\, d\tau, \end{aligned} \tag{10}$$

where $x_2(t)$ is the solution of the equation $dx/dt = f(t, x)$ with the initial condition $x_2(\theta_i) = x_1(\theta_i +)$.

From equality (9) there follows that for the proof of the theorem it is sufficient to show that the trivial solution of Eqs. (10) is asymptotically stable.

Applying the method of the proof of Theorem 3, one can verify that Eqs. (10) are equivalent to the system

$$du/dt = A(t)u + f(t, u), t \neq \tau_i, \quad \Delta u|_{t=\tau_i} = P_i u + J_i(u), \tag{11}$$

where $\|f(t, u)\|/\|u\| \to 0$, $\|J_i(u)\|/\|u\| \to 0$ for $\|u\| \to 0$.

It is known that there exists a piecewise continuous, T-periodic Lyapunov transform [9], which reduces the system (11) to the equations

$$dv/dt = Cv + g(t, u), \quad t \neq \tau_i, \quad \Delta v|_{t=\tau_i} = V_i(v), \tag{12}$$

preserving the weak nonlinearity.

As follows from the assumptions of the theorem, all the eigenvalues of matrix C have negative real parts. From here it follows [4] that the zero solution of system (12) is asymptotically stable. The theorem is proved.

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