ALMOST-PERIODIC SOLUTIONS OF NONLINEAR IMPULSE SYSTEMS

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In the solution of many problems of nonlinear mechanics [1], the necessity arises to analyze discontinuous almost-periodic solutions of differential equations with impulse action.

In this article, a new definition is given of a discontinuous almost periodic function [2], and some of its properties are analyzed. For systems of differential equations with impulse equation on surfaces, the analogs of the theorems from [3, 4] are proved.

Let us denote by $\mathfrak{R}$ the set of piecewise-continuous vector-functions defined on the real axis R, expecting a countable set of points of discontinuity of the first kind (for different functions from $\mathfrak{R}$, the points of discontinuity do not necessarily coincide). The sets of points of discontinuity do not have finite limiting elements and are unbounded on the left and right.

Using the method of [5], let us determine in $\mathfrak{R}$ a distance function in the following manner. Functions f and g from $\mathfrak{R}$ we call ε-equivalent and denote $f \overset{\varepsilon}{\sim} g$, if a) the points of discontinuity of these functions can be enumerated $t_k^f, t_k^g, k \in Z$ (Z – is the set of integers), admitting finite multiplicity, in the order of position on the axis R so that $|t_k^f - t_k^g| < \varepsilon$; and b) there exist sequences of strictly increasing numbers $t_k', t_k'', k \in Z$, $t_k', t_k'' \to -\infty$ for $k \to -\infty$, $t_k', t_k'' \to +\infty$ for $k \to +\infty$, for which the relations $\sup\limits_{\substack{t \in (t_k', t_{k+1}') \\ t' \in (t_k'', t_{k+1}'')}} \|f(t) - g(t')\| < \varepsilon$, $|t_k' - t_k''| < \varepsilon$, $k \in Z$ are true.

<u>LEMMA 1.</u> Let the functions $\varphi_1, \varphi_2, \varphi_3$ belong to the set $\mathfrak{R}$ and $\varphi_1 \overset{\varepsilon_1}{\sim} \varphi_2, \varphi_2 \overset{\varepsilon_2}{\sim} \varphi_3$. Then $\varphi_1 \overset{\varepsilon_1+\varepsilon_2}{\sim} \varphi_3$.

<u>Proof.</u> Let $t_k^1, t_k^2, k \in Z$, be sequences of points of discontinuity, respectively, of the functions φ_1 and φ_2, and t_k^3, t_k' be sequences of points of discontinuity of φ_3 such that $|t_k^1 - t_k^3| < \varepsilon_1$, $|t_k' - t_k^2| < \varepsilon_2$.

One can obtain a sequence t_k' from t_k^3, by implementing successively the following transformations: 1) a change of the multiplicity of some elements of the sequence; 2) a shift of the numbers of all the elements by one and the same integer. Precisely, the same such transformations can be applied to the sequence t_k^1. Therefore, we will consider that the sequences of points of discontinuity of $\varphi_1, \varphi_2, \varphi_3$ can be enumerated t_k^1, t_k^2, t_k' so that $|t_k^1 - t_k'| < \varepsilon_1$ $|t_k^2 - t_k'| < \varepsilon_2$. It, hence, follows that $|t_k^1 - t_k^2| < \varepsilon_1 + \varepsilon_2$, $k \in Z$.

Let t_k, t_k', t_k'', t_k^* be sequences of points for which

$$\sup_{\substack{t \in (t_k, t_{k+1}) \\ t' \in (t_k'', t_{k+1}'')}} \|\varphi_1(t) - \varphi_2(t')\| < \varepsilon_1 \qquad \sup_{\substack{t \in (t_k^*, t_{k+1}^*) \\ t' \in (t_k', t_{k+1}')}} \|\varphi_2(t) - \varphi_3(t')\| < \varepsilon_2,$$

$|t_k - t_k''| < \varepsilon_1$, $|t_k' - t_k^*| < \varepsilon_2$, $t_k, t_k' \to +\infty$ for $k \to +\infty$, $t_k, t_k' \to -\infty$ for $k \to -\infty$.

Let us unite the sets $\{t_k''\}$ and $\{t_k^*\}$ and enumerate the elements of the set obtained with multiplicity equal to unity, by the integers. Let us denote the sequence obtained $\{t_k''\}$. Let us supplement the sets $\{t_k\}$ and $\{t_k'\}$ by the elements from the sets $\{t_k''\}/\{t_k\}$ and $\{t_k''\}/\{t_k'\}$, respectively. One can enumerate the sets obtained with multiplicity equal to unity, in such a manner (denoting them as previously $\{t_k\}$ and $\{t_k'\}$) that $|t_k - t_k''| < \varepsilon_1$, $|t_k' - t_k''| < \varepsilon_2$ and

$$\sup_{\substack{t \in (t_k, t_{k+1}) \\ t' \in (t_k'', t_{k+1}'')}} \|\varphi_1(t) - \varphi_2(t)\|, \qquad \sup_{\substack{t \in (t_k'', t_{k+1}'') \\ t' \in (t_k', t_k')}} \|\varphi_2(t) - \varphi_3(t)\|.$$

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It follows from the last inequalities that for any integer k, we have the relations $\sup_{\substack{t\in(t_k,t_{k+1})\\ t'\in(t'_k,t'_{k+1})}} \|\varphi_1(t)-\varphi_3(t)\|<\varepsilon_1+\varepsilon_2$ and $|t'_k-t_k|<\varepsilon_1+\varepsilon_2$.

The lemma is proved.

Let the number $\rho(f,g)=\inf_{f\overset{\varepsilon}{\sim}g}\varepsilon$ define the distance between the functions f and g from $\mathfrak{R}$. Let us fix the function $\varphi\in\mathfrak{R}$ and denote by $\mathfrak{R}_\varphi$ the set of all functions f from $\mathfrak{R}$, for which the distance $\rho(f,\varphi)$ is a finite number. One can verify that then $\mathfrak{R}_\varphi$ is a metric space.

Let $I\doteq[a,b]$ be an interval from R. By $\mathfrak{R}_I$ let us denote the set of contractions of all functions from $\mathfrak{R}$ on I. Let us call functions f and g from $\mathfrak{R}_I$, ε-equivalent if: a) the points of discontinuity of these functions can be enumerated, allowing finite multiplicity, so that $|t_k^f-t_k^g|<\varepsilon$, where t_k^f and t_k^g are, respectively, points of discontinuity of f and g; and b) there exist points t'_k, t''_k, $k=0,1,\ldots,n$, $a=t'_0<t'_1<\ldots<t'_n=b$, $a=t''_0<t''_1<\ldots<t''_n=b$ such that $\sup_{\substack{t\in(t'_k,t'_{k+1})\\ t'\in(t''_k,t''_{k+1})}}\|f(t)-g(t')\|<\varepsilon$ and $|t'_k-t''_k|<\varepsilon$.

It is proved similarly to Lemma 1 that if the functions f, g, and h belong to the set $\mathfrak{R}_I$ and $f\overset{\varepsilon_1}{\sim}g, g\overset{\varepsilon_2}{\sim}h$, then $f\overset{\varepsilon_1+\varepsilon_2}{\sim}h$. It hence follows that if for any functions f and g from $\mathfrak{R}_I$, the distance between them is determined to equal $\rho(f,g)=\inf_{f\overset{\varepsilon}{\sim}g}\varepsilon$, then $\mathfrak{R}_I$ turns into a metric space.

Let $\omega\varphi(c,d)$ be an oscillation of the function φ on the interval (c, d).

<u>LEMMA 2.</u> Let the Cauchy sequence $\{f_n\}\subset\mathfrak{R}_I$ satisfy the conditions: a) the union U of the sets of points of discontinuity of all the functions f_n has in I a finite number of limit elements; and b) for any ε > 0 there exists δ > 0 such that for any $t\in I$ and f_n, and relations $\omega f_n(a,a+\delta)<\varepsilon$, $\omega f_n(b-\delta,b)<\varepsilon$, $\omega^* f_n(t,t+\delta)<\varepsilon$, are valid, where $\omega^* f(\theta,\theta')=\inf_t\max\{\omega f(\theta,t), \omega f(t,\theta')\}$.

Then the sequence f_n converges to a function, the contraction of which on the set I without the points of discontinuity of this function belongs to $\mathfrak{R}_I$.

<u>Proof.</u> Let the number of limit points of the set U be finite and equal p. It is assumed that a point is a limit for the set U too in that case when it is a point of discontinuity for an infinite number of functions f_n. Let us denote the limit points $t_1, t_2, \ldots, t_p$. Without destroying generality, one can consider that for any sufficiently small number ε > 0 there exists a sufficiently large number, beginning with which, all the functions f_n are continuous on $[t_i+\varepsilon, t_{i+1}-\varepsilon]$ and, consequently, uniformly converge on this interval. Actually, let $t_k^n, t_k^{n+r}, t_{k+1}^n, t_{k+1}^{n+r}$, $n>0$, r > 0 be the points of division, determined by the equivalence of the functions f_n, f_{n+r} and such that $|t_k^n-t_k^{n+r}|<\delta$, $|t_{k+1}^n-t_{k+1}^{n+r}|<\delta$ and for any $\theta\in(t_k^n,t_{k+1}^n)$, $\theta'\in(t_k^{n+r},t_{k+1}^{n+r})$ the inequality $\|f_{n+r}(\theta')-f_n(\theta)\|<\delta$ is satisfied. Let ε' > 0 be arbitrarily small. Let us choose δ > 0 so that due to the equicontinuity of the functions f_n and f_{n+r} for any t' and t" from the interval $[t_i+\varepsilon, t_{i+1}-\varepsilon]$, the relations $\|f_n(t')-f_n(t'')\|<\varepsilon'$, $\|f_{n+r}(t')-f_{n+r}(t')\|<\varepsilon'$, are satisfied only if $|t'-t''|<\delta$. Then, for any $t\in[t_i+\varepsilon, t_{i+1}-\varepsilon]$ the inequality $\|f_{n+r}(t)-f_n(t)\|<\varepsilon'+\delta$ is true.

The continuity on this interval of the limit function f(t) follows from the uniform convergence of the functions f_n on the interval $[t_i+\varepsilon, t_{i+1}-\varepsilon]$. Since ε is arbitrarily small, then f is a function that is continuous on the interval (t_i, t_{i+1}). Again, using the uniform convergence of the sequence f_n and equicontinuity, we find that f has finite limit values at t_{i+1} on the left, and at t_i on the right.

The arguments, conducted for (t_i, t_{i+1}), are true for all the remaining intervals (a,t_1), (t_n,b), (t_j,t_{j+1}), j = 1, 2,...,n − 1. The lemma is proved.

With the help of Lemma 2, the next lemma is validated.

<u>LEMMA 3.</u> In order that the set $M\subset\mathfrak{R}_I$ be compact in $\mathfrak{R}_I$, it is enough that the following conditions be satisfied: a) the union U of the sets of points of discontinuity of all functions from M has a finite number of limit elements: and b) for any ε > 0 there exists δ > 0 such that for any $f\in M$ and $t\in I$ the inequalities $\omega f(a,a+\delta)<\varepsilon$, $\omega f(b-\varepsilon,b)<\varepsilon$, $\omega^* f(t-\delta,t)<\delta$ are true; and c) the function s from M are bounded in the aggregate.

Definition. Let us call a function $\varphi \in \mathfrak{R}$ almost periodic (a.p.) if for any $\varepsilon > 0$ there is a relatively dense set of real numbers τ, for each element τ of which the inequality $\rho(\varphi(t+\tau), \varphi(t)) < \varepsilon$ holds.

The same as for almost periodic functions in the sense of Bohr, it is verified that each a.p. function is bounded and for each $\varepsilon > 0$ there is $\delta > 0$ satisfying the condition: for all t from the domain of definition of an a.p. function φ, the relation $\omega^*\varphi(t-\delta, t) < \varepsilon$ is satisfied.

Let us enumerate the points of discontinuity of φ with a multiplicity equal to unity. For any integers i and j let $\bar{t}_i^j = t_{i+j} - t_i$, and let us construct for each j the sequence $\bar{t}_i^j$, $i \in Z$.

LEMMA 4. Let the sequence t_i of points of discontinuity of an a.p. function φ be such that $\inf_i \bar{t}_i^1 > 0$. Then, the sequences $\bar{t}_i^j$ are equicontinuously almost periodic (e.a.p.) in i.

Proof. Let τ be an ε-almost period of $\varphi(t)$. Let us show that the points of discontinuity of $\varphi(t+\tau)$ can be enumerated t_i' in such a manner that $t_i' = t_{i+j_0} - \tau$ and $|\bar{t}_i^{j_0} - \tau| < \varepsilon$, where j_0 is an integer, common for all $i \in Z$. One can consider that $2\varepsilon < \theta$ and therefore for arbitrary $i \in Z$ there exists a unique point t_{i+j_0} such that the point $t_i - \tau$ is in an ε-neighborhood of the point t_{i+j_0}. Furthermore, the neighboring point t_{i+j_0+1} in its ε-neighborhood contains a unique point $t_{i+1} - \tau$. Now, the assertion follows from the arbitrariness of i.

Since the set of real numbers τ satisfying the inequality $|\bar{t}_i^j - \tau| < \varepsilon$ for any $i \in Z$, is relatively dense, then in accord with Lemma 3 [6, p. 285], the sequences $\bar{t}_i^j$ are e.a.p. The lemma is proved.

THEOREM 1. In order that the function $\varphi \in \mathfrak{R}$ be almost periodic, it is sufficient in the case when $\inf \bar{t}_i^1 > 0$, and necessary that each sequence of shifts $\varphi(t+h_n)$ be compact in $\mathfrak{R}_\varphi$.

Proof. Necessity. Let φ be an a.p. function $\inf \bar{t}_i^1 = \theta > 0$. Let us construct a sequence ε_n, $\varepsilon_n > 0$, $\varepsilon_n \to 0$. Let us denote by $l_n = l(\varepsilon_n)$ the indices of the density of the sets of ε_n-almost periods of φ. Let us select a sequence of intervals $I_n = [-\varepsilon_n - l_n/2, \varepsilon_n + l_n/2]$. Let us assume that t_1^1 is the (first in I_1) point of discontinuity of $\varphi(t+h_1)$, and t_1^2 is the (first on the left in I_1) point of discontinuity of $\varphi(t+h_2)$, etc. By the Bolzano–Weierstrass theorem, the sequence t_1^n has a subsequence $t_1^{n_1}$ converging to the limit point t_1. Now let $t_2^{n_1}$ be a sequence of the second (on the left in I_1) points of discontinuity of the functions $\varphi(t+h_{n_1})$. Let us select from $t_2^{n_1}$ a subsequence $t_2^{n_2}$ converging to t_2, etc.

Since the distance between neighboring points of discontinuity is not less than θ, then after a finite number p of steps, we obtain a sequence $\varphi(t+h_{n_p})$, and the union of the sets of points of discontinuity of all the functions, which has a finite number of limit elements, that is, this sequence satisfies for $\mathfrak{R}_{I_1}$ condition a) of Lemma 3. The validity of conditions b) and c) follow from the property of boundedness and uniform continuity on the totality of intervals not containing points of discontinuity of an a.p. function. In such a manner, one can select from $\varphi(t+h_n)$ a subsequence $\varphi(t+h_n^1)$ converging in $\mathfrak{R}_{I_1}$. Similarly, from $\varphi(t+h_n^1)$ is extracted a subsequence $\varphi(t+h_n^2)$ converging in $\mathfrak{R}_{I_2}$, etc. The diagonal sequence $\varphi(t+h_n^n)$ converges in each of the spaces $\mathfrak{R}_{I_n}$ to $\varphi_0 \in \mathfrak{R}$. For simplicity we will suppose that the sequence $\varphi(t+h_n)$ itself possesses such a property.

Let $\varepsilon > 0$ be arbitrary. For any k, there is a number N_k starting with which, for any natural number ℓ, the inequality $\rho_{I_k}(\varphi_{n+l}(t+h_{n+l}), \varphi_n(t+h_n)) < \varepsilon$ is true. Due to the ε_n-almost periodicity, it follows that for $n > N_k$ the inequality $\rho(\varphi_{n+l}(t+h_{n+l}), \varphi_n(t+h_n)) < \varepsilon + \varepsilon_n$ is satisfied. It hence follows that the sequence $\varphi(t+h_n)$ converges in $\mathfrak{R}_\varphi$ to $\varphi_0(t)$. One can show that this function φ_0 is almost periodic.

Sufficiency. Let us assume that φ is not an a.p. function. Then, there exists $\varepsilon_0 > 0$ and a sequence of intervals $[h_n - l_n, h_n + l_n]$, $l_n \geqslant \max_{m<n} |h_m|$, where ℓ_1 is arbitrary, for each ζ from $\rho(\varphi(t+\zeta), \varphi(t)) \geqslant \varepsilon_0$ is true.

Let us consider the sequence of shifts $\varphi(t+h_n)$. For each subsequence $\varphi(t+h_{n_k})$ of it, from the fact that for $n_p > n_k$, we have $h_{n_p} - h_{n_k} \in [h_{n_p} - l_{n_p}, h_{n_p} + l_{n_p}]$, it follows that $\rho(\varphi(t+h_{n_p}), \varphi(t+h_{n_k})) = \rho(\varphi(t+(h_{n_p} - h_{n_k}), \varphi(t)) \geqslant \varepsilon_0$. This means that the sequence $\varphi(t+h_{n_k})$ is not converging in $\mathfrak{R}_\varphi$. This contradicts the proposition. The theorem is proved.

Let the elements of the space $\mathfrak{D}$ be countable sets of real numbers, unbounded on the left and right. We will call the elements $M_1, M_2 \in \mathfrak{D}$, ε-equivalent and denote $M_1 \overset{\varepsilon}{\sim} M_2$ if the points of these sets can be enumerated by the integers, respectively, m_i^1, m_i^2, allowing finite multiplicity, in the order of position on the axis R so that $\sup_i |m_i^1 - m_i^2| < \varepsilon$.

We will consider the distance between M_1 and M_2 in $\mathfrak{D}$ to be the number $\rho_{\mathfrak{D}}(M_1, M_2) = \inf_{M_1 \overset{\varepsilon}{\sim} M_2} \varepsilon$.

Let the distance between any neighboring points of set T be not less than some positive number. Let us enumerate the elements of the set T with multiplicity equal to unity. Let $T = \{t_i\}$.

One can verify that the following lemma is true.

LEMMA 5. In order that the sequences $\bar{t}_i^j$ be e.a.p., it is necessary and sufficient that from each infinite sequence of shifts $\{\{t_i + h_n\}_i\}_n$, one can select a subsequence converging in $\mathfrak{D}$.

Let us denote by $\mathfrak{R}(K)$ the space of bounded functions f(t, x), continuous in x, on the compactum $K \subset R^n$, allowing if the point (t, x) falls on the surface $t = t_i(x)$, $i \in Z$, discontinuities of the first kind, in such a manner that for fixed x, the function $f(t, x) \in \mathfrak{R}$. We define the distance in $\mathfrak{R}(K)$ to equal the number $\rho_K(f_1, f_2) = \sup_K \rho(f_1(t, x), f_2(t, x))$, where $f_1, f_2 \in \mathfrak{R}(K)$.

Let $\mathfrak{M}(K)$ be the space of sequences of vectors $\{I_i(x)\}$, bounded on K, each element of which is continuous in x. The distance between the sequences $I_i^{(1)}(x)$ and $I_i^{(2)}(x)$ from $\mathfrak{M}(K)$ equals the number $\sup_{Z \times K} \| I_i^{(1)} - I_i^{(2)} \|$.

Also let $\mathfrak{D}(K)$ be the space of sequences $t_i(x)$, $i \in Z$, of continuous functions $t_i(x): R^n \to R$ with the topology determined by the distance $\sup_K \rho_{\mathfrak{D}}(t_i^{(1)}(x), t_i^{(2)}(x))$, where $t_i^{(1)}(x)$ and $t_i^{(2)}(x)$ are sequences from $\mathfrak{D}(K)$.

Let us denote by $\mathfrak{U}(K)$ the topological space-product $\mathfrak{R}(K) \times \mathfrak{M}(K) \times \mathfrak{D}(K)$.

Let us consider a system of differential equations with the impulse action

$$dx/dt = f(t, x),\ t \neq t_i(x), \quad \Delta x|_{t=t_i(x)} = I_i(x), \tag{1}$$

in which $x \in R^n$, $\Delta x|_{t=t_i} = x(t_i +) - x(t_i -)$, and the function f(t, x) and the sequences $I_i(x)$, $t_i(x)$ are such that for any compactum $K \subset R^n$ we have $(f(t, x), I_i(x), t_i(x)) = B_0 \in \mathfrak{U}(K)$. The sequence $t_i(x)$ is such that

$$\inf_i [\inf_K t_i(x) - \sup_K t_{i-1}(x)] > 0 \tag{2}$$

and the sequences $\bar{t}_i^j(x)$ are e.a.p. in i, uniformly with respect to $x \in K$.

We will consider that each solution of the system (1) meets with each of the surfaces $t = t_i(x)$ not more than one time. For this, for example, it is sufficient to assume that the inequalities (determined in [7]) $t_i(x + I_i(x)) \leqslant t_i(x)$, $|t_i(x) - t_i(y)| \leqslant k \|x - y\|$, $kM < 1$ are true for all $i \in Z$, $x, y \in K$, where $M = \sup_{R \times K} \|f(t, x)\|$.

Let us assume that f(t, x) is a function a.p. in t in the metric of the space $\mathfrak{R}(K)$, and $I_i(x)$ is a sequence a.p. in i in the metric of the space $\mathfrak{M}(K)$.

For simplicity, let us identify system (1) with the element B_0. One can verify that if x(t) is a solution of equation B_0, then the function $x(t + \theta)$, where θ is a solution of the system $dx/dt = f(t + \theta, x), i \neq t_i(x) - \theta, \Delta x|_{t=t_i(x)-\theta} = I_i(x)$ which we will identify with $B_\theta \in \mathfrak{U}(K)$. Let h_n be an arbitrary sequence of real numbers.

Using successively Theorem 1 and Lemma 5, let us extract from it a subsequence h_{n_k}, for which the sequence $f(t + h_{n_k})$ converges in $\mathfrak{R}(K)$ to a function a.p. in t, and the sequence $\{t_i(x) - h_{n_k}\}_k$ converges in $\mathfrak{D}(K)$ to a sequence $t_i^h(x)$ for which the sequences $\bar{t}_i^{hj}(x)$ are e.a.p. in i.

Let i_{n_k} be a sequence of integers for which, in agreement with Lemma 5, the sequences $t_{i+i_{n_k}}(x)$, $i \in Z$, converge uniformly in i as $k \to +\infty$ to $t_i^h(x)$. Let us consider the sequence of shifts $I_{i+i_{n_k}}(x)$. One can select from it a subsequence, converging to $I_i^h(x)$. The sequence I_i^h is a.p. in i. In this manner, any sequence of shifts B_{h_n} of B_0 is compact in $\mathfrak{U}(K)$ in the sense of convergence defined above.

Let us denote by H the closure of the shifts B_θ, $\theta \in R$, in $\mathfrak{U}(K)$. Let $B^h \in H$ and H^h be the closure of the shifts B_θ^h, $\theta \in R$. Similarly to the proof of the corresponding lemma for ordinary differential equations, one can show that $H^h = H$.

The bounded solution $x_0(t)$ of the system of equations from H is called divided in $R \times K$ if it is unique in K, or for any other bounded solution x(t), the values of which are contained in K, for all t from the common domain of definition of the functions x_0 and x, the inequality $\| x(t) - x_0(t) \| \geqslant p$ is true, where p is a positive constant, depending only on $x_0(t)$.

If the system B_0 has a bounded solution, contained in K, and all the bounded solutions of the systems from H, contained in K, are divided, then from the equation $H^h = H$ for any h, it follows that all the equations from H have an identical finite number of bounded solutions, the values of which are located in K, and there exists a constant $p > 0$ such that for any two bounded solutions x(t) and y(t) of one equation from H, the inequality

$$\inf_{t \in R'} \| x(t) - y(t) \| \geqslant p, \tag{3}$$

is true, where R' is the intersection of the domains of definition of the solutions x(t) and y(t).

THEOREM 2. Let the system B_0 have a bounded solution, the values of which are contained in K, and all the bounded solutions of any equation from H, contained in K, are divided.

Then all these bounded solutions are almost periodic.

Proof. If one assumes that if the conditions of the theorem are satisfied, some bounded solution of $x = \xi(t)$ of B_0 is not almost-periodic, then there is a sequence of shifts $\xi(t + h_n)$, which for any $I \subset R$ converges in $\mathfrak{R}_I$, but is not convergent in $\mathfrak{R}_\xi$.

One can select from h_n, subsequences h_{p_k} and h_{m_k}, for which there exists a sequence of intervals I_k, $I_k \subset I_{k+1}$, $\bigcup_k I_k = R$ such that

$$\gamma < \rho_{I_k}(\xi(t + h_{p_k}),\ \ \xi(t + h_{m_k})) < p/2, \tag{4}$$

where γ is some positive constant. It follows from (4) that $\xi(t + h_{p_k})$ and $\xi(t + h_{m_k})$ converge, respectively, in each space $\mathfrak{R}_I$ to different bounded solutions η_1 and η_2 of one and the same equation from H and the inequality $\gamma < \rho(\eta_1(t), \eta_2(t)) < p/2$ is true, contradicting condition (3). The theorem is proved.

It follows from Theorem 2, in particular, that if each equation from H has a unique bounded solution, then it is almost periodic.

LITERATURE CITED

1. N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations [in Russian], Fizmatgiz, Moscow (1963).
2. A. M. Samoilenko, N. A. Perestyuk, and M. U. Akhmetov, Almost Periodic Solutions of Differential Equations with Impulse Action [in Russian], Preprint 83.26, Math. Inst., Akad. Nauk Ukr. SSR (1983).
3. L. Amerio, "Soluzioni quasi periodiche o limate, di sistemi differenziali non lineri quasi-periodici o limitati," Ann. Math. Pura Appl., 39, 97-119 (1955).
4. B. M. Levitan and V. V. Zhikov, Almost Periodic Functions and Differential Equations [in Russian], Moscow State Univ. (1978).
5. A. N. Kolmogorov, "On the Skorokhod convergence," Teor. Veroyatn. Primen., 1, No. 2, 239-247 (1956).
6. A. Halanay and D. Wexler, The Qualitative Theory of Systems with Impulse [in Rumanian], Editura Academiei Republicii Socialiste Romania, Bucuresti (1968) [Russian translation, Moscow, Mir (1971)].
7. S. I. Gurgula and N. A. Perestyuk, "Lyapunov's second method in systems with impulse action," Dokl. Akad. Nauk Ukr. SSR, Ser. A, 10, 11-14 (1982).