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Control and optimal response problems for quasilinear impulsive integrodifferential equations

M.U. Akhmet [a,d,*], M. Kirane [b], M.A. Tleubergenova [c], G.W. Weber [d]

[a] Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey
[b] Laboratoire de Mathematiques, Universite de la Rochelle, Av. M. Crepeau, 17042 La Rochelle Cedex, France
[c] Branch of K. Satpaev Kazakh National Technical University, Mares'eva str., 10, 463000 Aktobe, Kazakhstan
[d] Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey

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Abstract

In various real-world applications, there is a necessity given to steer processes in time. More and more it becomes acknowledged in science and engineering, that these processes exhibit discontinuities. Our paper on theory of control (especially, optimal control) and on theory of dynamical systems gives a contribution to this natural or technical fact.

One of the central results of our paper is the Pontryagin maximum principle [L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, John Wiley, New York, 1962] which is considered in sufficient form for the linear case of impulsive differential equations. The problem of controllability of boundary-value problems for quasilinear impulsive system of integrodifferential equations is investigated. The control consists of a piecewise continuous function part as well as impulses which act at a variable time.

By studying the optimal control of response, we give a first inclusion of an objective function. By this pioneering contribution, we invite to future research in the wide field of optimal control with impulses and in modern challenging applications.
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* Corresponding author. Tel.: +90 3122105355; fax: +90 3122101282.
E-mail address: marat@arf.math.metu.edu.tr (M.U. Akhmet).

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1. Introduction

The fast scientific development in the foundations and microworld of biology has led to a reconsideration about nature, about some characteristics of life. In fact, scientists agree to its continuous nature enriched by discretely arising discontinuities. The latter ones are also called *jumps* or, more from the viewpoint of energy, *impulses*. This combined *continuous-discrete* character of life does not only mean a theoretical insight into the "miracle of life", but it is also important for a variety of applications in bioinformatics and practical utilizations in biotechnologies. Genetic or metabolic processes need to be well understood, e.g., for medical purposes, biochemical processes need to well guided, e.g., in drug design and pharmacy. (For a small impression and information in proceeding research, also concerning many further references, we refer to [1,11,17–20].) The acknowledgement of this practical importance has made optimization of biosystems and bioinformatics to become a part also of modern *operational research* in its international conferences and journals. Furthermore, jumps or impulses not only arise on the micro level but also on a more global and macro level of computational biosciences, for example, in gene dynamics and in population dynamics. Herewith, jumps and impulses are a characteristic feature throughout in computational biology. Today, this scientific situation and landscape briefly sketched requests both (i) investigations in natural sciences in order more find and represent the threshold phenomena underlying the jumps, and (ii) research in mathematics in order to understand and optimize the biological processes with impulses includes. In both directions, (i) and (ii), researchers are, at least partially, standing at the very beginning. This paper is a contribution to that second scientific goal, to an opening and widening of optimal control theory for different kinds of impulsive dynamical systems, and vice versa, with various practical applications in live sciences, including medicine, in technology and social sciences. We mention that control and, in particular, optimal control also have further motivations coming from outside of biology, e.g., from information theory and processing and from chemical engineering.

In problems from optimal control theory, the objective functional to be minimized, may stand for, e.g., costs, energy, risk or variance. The wide theory of these problems, i.e., questions of feasibility and controllability, of optimality and stability, is very well investigated (cf., e.g., [11,14,22,31–34,37]). The optimal control problems studied in our pioneering paper, called *optimal response problems*, belong to the class of *time-minimal* or *terminal* control problems (see, e.g., [25–28,37]). In this paper, we are concentrating on a *linear* problem such that the optimization problem coming from necessary optimality conditions, called maximum principle, leads to a problem from linear programming, to be more precise, to a parametric family of such problems in the variable u. The generalization the to *non*linear case where also the surfaces of jumps may become generalized in addition, are recommended for future research.

The theory of impulsive differential equations is emerging as an important area of investigation since it is richer in problems in comparison with the corresponding theory of differential equations. Actually, many mathematical problems, e.g., dynamical and optimization ones, encountered in studying impulsive differential equations cannot be treated with the usual techniques of ordinary differential equations [21,30,35]. Here, we also mention and recall biological applications in population dynamics and genetics (mutation, experiments, etc. [39]) where impulses (jumps) arise naturally or are caused by control. Concerning jumps, thresholds and other combined continuous-discrete items we refer to [10,15,29]. Moreover, impulsive differential systems present a natural framework for mathematical modeling of several real-world problems [8,11,12,20,21,30,35]. However, the theory of integrodifferential equations with impulse actions on surfaces is not yet sufficiently elaborated compared to that of impulsive differential equations and integrodifferential equations. There are also several problems on controllability for impulsive systems which are connected with the results of the theory of integral and integrodifferential equations and have not been well investigated yet [3,6,7,9,13,24,31,36]. One of the original approaches to optimization problems is through a boundary-value problem. This method requires the development of the controllability problem of solutions for impulsive equations, particularly of integrodifferential systems of the form

$$dx/dt = A(t)x + \int_\alpha^t K(t,s)x(s)\,ds + C(t)u(t) + f(t) + \mu g(t,x,u,\mu), \quad t \neq \theta_i + \mu\tau_i(x,\mu),$$

$$\Delta x(\zeta_i) = B_i x(\zeta_i) + \sum_{j:\alpha<\zeta_j\leqslant\zeta_i} D_{ij}x(\zeta_j) + \int_\alpha^{\zeta_i} M_i(s)x(s)\,ds + Q_i v_i + J_i + \mu W_i(x(\zeta_i), v_i, \mu),$$

$$i = 1, 2, \ldots, p, \tag{1}$$

$$x(\alpha) = a, \quad x(\beta) = b. \tag{2}$$

Here, $\mu > 0$ is a small parameter, α, β, θ_i, $\zeta_i \in R$ and are such that $\alpha < \theta_1 < \cdots < \theta_p < \beta$, and $\zeta_i = \theta_i + \mu\varpi$ $\tau_i(x(\zeta_i), \mu)$; A, K, M_i, D_{ij} and B_i are $n \times n$ matrices; C and Q_i are $n \times m$ matrices; x, f, g, J_i, W_i, a, and b are n-vectors; u and v are m-vectors; $\tau_i(x, \mu)$, $i = 1, 2, \ldots, p$, are real-valued scalar functions; $\Delta x(t) \equiv x(t+) - x(t-)$, where $x(t+) = \lim_{h\to 0^+} x(t+h)$ and $x(t-) = \lim_{h\to 0^-} x(t+h)$. We assume that solutions are left continuous and therefore write $\Delta x(t) = x(t+) - x(t)$. In this paper, using some results from [2,4,6], we shall investigate the problem of controllability of boundary-value problems for quasilinear impulsive system of integrodifferential equations of the form (1), (2). We obtain our results by comparing solutions of integrodifferential equations having impulse actions at variable moments with solutions of integrodifferential equations having impulse actions at fixed moments. This comparison method was proposed by Akhmetov and Perestyuk in [2]. As being well known, the solutions of differential equations with variable moments of impulse effect may experience pulse phenomena, namely, they may hit a given surface of discontinuity a finite or infinite number of times causing rhythmical beating [30,35]. This results in additional complications in studying such systems and, therefore, in most cases it is necessary to find conditions that guarantee the absence of beating. In the present article, we also provide a new condition for the absence of beating (see Theorem 5) which is based on the method of small parameter. Finally, a *maximum principle of Pontryagin type* is proposed for the time-optimal problem in linear case. The results of this paper may be considered as a continuation or a generalization of the results obtained in [4–6], where linear and quasilinear impulsive differential systems were considered. The results can be useful and an invitation for investigating problems of optimum control for discontinuous dynamics in general [8,34].

2. Preliminaries

In what follows, we denote by PAC[α, β] the set of all functions $x:[\alpha, \beta] \to R^n$ which are piecewise absolutely continuous and continuous on the left with discontinuities of the first kind at points θ_i, $i = 1, 2, \ldots, p$. Denote, next, by $L_2^r[\alpha, \beta]$ the set of all square integrable functions $\phi:[\alpha, \beta] \to R^r$ and by $D^r[1, p]$ the set of all finite sequences $\{\xi_i\}$, $\xi_i \in R^r$, $i = 1, \ldots, p$, where p and r are fixed positive integers. Furthermore, we define $\Pi^r[\alpha, \beta] := L_2^r[\alpha, \beta] \times D^r[1]$ and identify its elements as $\{\phi, \xi\}$, and let

$$\langle\{\phi, \xi\}, \{\omega, v\}\rangle = \int_\alpha^\beta (\phi, \omega)\,dt + \sum_{i=1}^p (\xi_i, v_i)$$

be an inner product in $\Pi^r[\alpha, \beta]$, where (,) is the Euclidean scalar product in R^r. Let us introduce the norm $\|\{\phi, \xi\}\|_{[\alpha,\beta]} = \langle\{\phi, \xi\}, \{\phi, \xi\}\rangle^{\frac{1}{2}}$ in $\Pi^r[\alpha, \beta]$. Throughout this paper we need the following conditions:

(C1) the functions g, W_i, τ_i, $i = 1, 2, \ldots, p$, are continuous with respect to their variables and continuously differentiable with respect to x, u, and v;
(C2) the matrix $K(t, s):[\alpha, \beta] \times [\alpha, \beta] \to R^n \times R^n$ is square integrable;
(C3) the columns of the matrices $A(t)$ and $M_i(t)$, $i = 1, 2, \ldots, p$, are in $L_2^n[\alpha, \beta]$;
(C4) $\{f, J\} \in \Pi^n[\alpha, \beta]$;
(C5) $\det(I + B_j + D_{jj}) \neq 0$, $\det(I + B_j) \neq 0$ for $j = 1, 2, \ldots, p$.

The process defined by (1) for fixed μ and $\{u, v\}$ operates as follows: The point $P_t(t, x(t))$, starting at (α, a), moves along the curve defined by the solution $x(t) = x(t, \alpha, a)$ of the equation

$$\frac{dx}{dt} = A(t)x(t) + \int_\alpha^t K(t,s)x(s)\,ds + C(t)u + f(t) + \mu g(t,x,u,\mu). \tag{3}$$

The motion along this curve terminates at time $t = \zeta_1$, when the point P_t arrives at the surfaces of discontinuity so that $\zeta_1 = \theta_1 + \mu\tau_1(x(\zeta_1), \mu)$. At that moment the point P_t performs a jump

$$\Delta x|_{t=\zeta_1} = B_1x(\zeta_1) + D_{11}x(\zeta_1) + \int_\alpha^{\zeta_1} M_1(s)x(s)\,ds + Q_1v_1 + J_1 + \mu W_1(x(\zeta_1), v_1, \mu)$$

and proceeds to move along the curve described by the solution $x(t, \zeta_1, x(\zeta_1+))$ of system (3), until it meets the next surface of discontinuity, and so on. We should note that each solution of (1) is a piecewise continuous with discontinuities of the first kind function.

Definition 2.1. Problem (1), (2), which we denote by $\Sigma_\mu(G)$, is said to be solvable for a given bounded set $G = G_a \times G_b \subset R^n \times R^n$ if there exists a positive real number μ_0, $\mu_0 = \mu_0(G)$, such that for all arbitrary $a, b \in G_a \times G_b$ and $\mu < \mu_0$ there is a control $\{u, v\} \in \Pi^m$ for which system (1) admits a solution $x(t)$ satisfying (2).

Let s be a positive real number, and let T_s be the subset of elements (x, u, v) satisfying the inequality $|x| + |u| + |v| \leqslant s$, where $|\cdot|$ is the Euclidean norm in R^n.

For a fixed positive real number μ_1, we define

$$G_s = \{(x,u,v,t,i,\mu) : (x,u,v) \in T_s,\ \alpha \leqslant t \leqslant \beta,\ i = 1,2,\ldots,p,\ 0 < \mu \leqslant \mu_1\}.$$

Let a positive real number H be fixed and

$$m_1 = \max\left\{\sup_t |A(t)|, \sup_t |C(t)|, \sup_{t,s} |K(t,s)|, \sup_{i,t} |M_i(t)|, \max_i |B_i|, \max_{ij} |D_{ij}|\right\},$$

$$m_2 = \max\left\{\sup_t |f(t)|, \max_i |J_i|\right\},$$

$$m_3 = \max\left\{\max_{(x,u,t,\mu)\in \mathrm{pr}_{(1,2,4,6)}(G_H)} |g|, \max_{(x,v,i,\mu)\in \mathrm{pr}_{(1,3,5,6)}(G_H)} |W_i|, \max_{(x,i,\mu)\in \mathrm{pr}_{(1,5,6)}(G_H)} |\tau_i|\right\},$$

where the set notation $\mathrm{pr}_{(1,2,4,6)}(G_H)$ means the natural projection of the set G_H of points (6-tuples) (x, u, v, t, i, μ) with respect to the Cartesian coordinates $(\tilde{x}, \tilde{u}, \tilde{t}, \tilde{\mu})$ (in the tuple sense: here, the 4-tuple's components are enumerated by 1, 2, 4 and 6). The notations $\mathrm{pr}_{(1,3,5,6)}(G_H)$ and $\mathrm{pr}_{(1,5,6)}(G_H)$ are analogously understood; here, i may be a discrete variable.

It is not very difficult to observe in view of (C1) that there is a positive real number L such that

$$|g(t,x_1,u_1,v^1,\mu) - g(t,x_2,u_2,v^2,\mu)| \leqslant L\{|x_1 - x_2| + |u_1 - u_2| + |v^1 - v^2|\},$$

$$|W_i(x_1,v^1,\mu) - W_i(x_2,v^2,\mu)| \leqslant L\{|x_1 - x_2| + |v^1 - v^2|\},$$

$$|\tau_i(x_1,\mu) - \tau_i(x_2,\mu)| \leqslant L|x_1 - x_2|,$$

uniformly for all t, x_1, x_2, u_1, u_2, v^1, v^2 in G_H.

Definition 2.2. If for $h > 0$ there exists a positive real number μ_0, $\mu_0 = \mu_0(h)$, such that if $\mu < \mu_0$ then for every given subset $G = \{(a, b) \,|\, |a| < h, |b| < h\} \subset R^n \times R^n$ the problem $\Sigma_\mu(G)$ is solvable, then we say that the problem $\Sigma_\mu(G)$ is solvable.

Let us fix a number $\mu_1 > 0$ such that

$$\mu_1 m_3 < \min\left\{\theta_1 - \alpha, \frac{1}{2}(\theta_2 - \theta_1), \ldots, \frac{1}{2}(\theta_p - \theta_{p-1}), \beta - \theta_p\right\}. \tag{4}$$

Lemma 1. *Assume that every solution $x(t)$ of (1) intersects every surface of discontinuity not more than once. If $\mu < \mu_1$, then every solution $x(t)$ of (1) which is in $pr_{(1)}(G_H)$ and is defined on $[\alpha, \beta]$ intersects each of the surface $t = \theta_i + \mu\tau_i(x, \mu)$, $i = 1, 2, \ldots, p$, exactly once.*

Proof. One can check that (4) implies

$$\max_{x\in \mathrm{pr}_1(G_H)} (\theta_i + \mu\tau_i(x,\mu)) < \min_{x\in \mathrm{pr}_1(G_H)} (\theta_{i+1} + \mu\tau_{i+1}(x,\mu)) \tag{5}$$

for all $i \in \mathrm{pr}_5(G_H)$ if $\mu < \mu_1$. Construct the following functions $\xi_i(t) = t - \theta_1 - \mu\tau_i(x(t),\mu), i = \overline{1,p}$. The conditions of our lemma imply that $\xi_i(\alpha) < 0, i = \overline{1,p}$. Assume, to the contrary, that the solution $x(t)$ is continuous on whole interval $[\alpha, \beta]$. Since $\xi_i(\beta) > 0, i = \overline{1,p}$, by Intermediate Value Theorem [23] there exists a first moment $t = \kappa_1$ of meeting for $x(t)$ with one of the surfaces. Using (5) one can show that the first intersection is with $t = \theta_1 + \mu\tau_1(x, \mu)$ and $\kappa_1 < \min_{x\in \mathrm{pr}_1(GH)}(\theta_2 + \mu\tau_2(x, \mu))$. Now, consider surfaces $t = \theta_i + \mu\tau_i(x,\mu), i = \overline{2,p}$, on the interval $[\kappa_1, \beta]$. Similarly to the previous case we can show that there exists a moment $t = \kappa_2$ of intersection of $x(t)$ with the surface $t = \theta_2 + \mu\tau_2(x, \mu)$. Continuing in the way, we can finish the proof. □

We will also need the following lemmas from [5] in the proof of our results. These two lemmas are analogous to Fubini's Theorem on changing the order of integration [24].

Lemma 2. *Let D_{ij}, $i, j = 1, 2, \ldots, p$, be constant matrices of size $n \times n$ and $\{\xi_i\} \in D^n[1, p]$. Then*

$$\sum_{i:\alpha<\theta_i<t}\ \sum_{j:\alpha<\theta_j\leqslant\theta_i} D_{ij}\xi_j = \sum_{i:\alpha<\theta_i<t}\ \sum_{i:\theta_i\leqslant\theta_j<t} D_{ji}\xi_i \quad \text{for each } t \in [\alpha, \beta].$$

Lemma 3. *Let $K(t, s)$ be a matrix of size $n \times n$. If $K(t, s)$ is square integrable with respect to s on $[\alpha, \beta]$ for each fixed $t \in [\alpha, \beta]$, and $\phi_i(t) \in L_2^n[\alpha, \beta]$ for $i = 1, 2, \ldots, n$, then*

$$\int_\alpha^t K(t,s) \sum_{i:\alpha<\theta_i<s} \phi_i(s)\,\mathrm{d}s = \sum_{i:\alpha<\theta_i<t} \int_{\theta_i}^t K(t,s)\phi_i(s)\,\mathrm{d}s \quad \textit{for each } t \in [\alpha, \beta]. \tag{6}$$

Let us consider an integral equation

$$x(t) = \int_\alpha^t G(t,s)x(s)\,\mathrm{d}s + \sum_{i:\alpha<\theta_i<t} S_i(t)x(\theta_i) + \sum_{i:\alpha<\theta_i<t} N_i(t)x(\theta_i+) + \sum_{i:\alpha<\theta_i<t} J_i + f(t) \tag{7}$$

under the following conditions:

(H1) the matrix $G(t, s) : [\alpha, \beta] \times [\alpha, \beta] \to R^n \times R^n$ is square integrable;
(H2) the columns of matrices $S_i(t)$ and $N_i(t)$ and the function $f(t)$ belong to PAC$[\alpha, \beta]$;
(H3) $\det(I - N_i(\theta_i+) + N_i(\theta_i)) \neq 0$ for $i = 1, 2, \ldots, p$.

Theorem 1. *Let conditions* (H1), (H2), *and* (H3) *hold. Then system (7) has a unique solution $x(t) \in PAC[\alpha, \beta]$ which can be represented as*

$$x(t) = \int_\alpha^t P_1(t,s)f(s)\,\mathrm{d}s + \sum_{i:\alpha<\theta_i<t} Q_i(t)J_i + \sum_{i:\alpha<\theta_i<t} P_2^i(t)f(\theta_i) + f(t) + \sum_{i:\alpha<\theta_i<t} J_i,$$

where $Q_i(t)$, $P_2^i(t)$, $i = 1, 2, \ldots, p$, and $P_1(t, s)$ are certain piecewise continuous function matrices of size $n \times n$.

Proof. Let $R(t, s)$ be the resolvent of the Volterra integral equation with kernel $G(t, s)$. In view of Lemma 3 it is easy to show that (7) is equivalent to

$$x(t) = \sum_{i:\alpha<\theta_i<t} \left[\int_{\theta_i}^t R(t,s)S_i(s)\,\mathrm{d}s + S_i(t)\right]x(\theta_i) + \sum_{i:\alpha<\theta_i<t} \left[\int_{\theta_i}^t R(t,s)N_i(s)\,\mathrm{d}s + N_i(t)\right]x(\theta_i+)$$
$$+ \sum_{i:\alpha<\theta_i<t} \left[\int_{\theta_i}^t R(t,s)\,\mathrm{d}s + I\right]J_i + \int_\alpha^t R(t,s)f(s)\,\mathrm{d}s + f(t), \tag{8}$$

where I is the $n \times n$ identity matrix.

Introducing the notation

$$S_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)S_i(s)\,\mathrm{d}s,$$
$$N_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)N_i(s)\,\mathrm{d}s,$$
$$p_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)\,\mathrm{d}s + I,$$

we obtain from (8) that

$$x(\theta_j) = \sum_{i:\alpha<\theta_i<\theta_j} [S_{ij}x(\theta_i) + N_{ij}x(\theta_i+)] + \sum_{i:\alpha<\theta_i<\theta_j} p_{ij}J_i + \int_\alpha^{\theta_j} R(\theta_j,s)f(s)\,\mathrm{d}s + f(\theta_i). \tag{9}$$

From (8) and (9) we also obtain that

$$x(\theta_j+) = (I - N_j(\theta_j+))^{-1}\Bigg\{(I + S_j(\theta_j+))x(\theta_i) + \sum_{i:\alpha<\theta_i<\theta_j} [S_i(\theta_j+) - S_i(\theta_j)]x(\theta_i) + \sum_{i:\alpha<\theta_i<\theta_j} [N_i(\theta_j+)$$
$$-N_i(\theta_j)]x(\theta_i+) + J_j + f(\theta_j+) - f(\theta_j)\Bigg\}. \tag{10}$$

The expressions (9) and (10) recursively determine $x(\theta_j)$ and $x(\theta_j+)$. Since the nonhomogeneous part of this system is a linear combination of vectors

$$\int_\alpha^{\theta_i} R(t,s)f(s)\,\mathrm{d}s, \quad f(\theta_i), \quad J_i, \quad \text{for } i = 1, 2\ldots, p,$$

$x(\theta_j)$ and $x(\theta_j+)$ can be written as linear combinations of these with certain matrix coefficients. The theorem is proved. □

3. Existence of solutions of integrodifferential equations

Now, we may state and prove a theorem on existence and uniqueness of solutions of the impulsive system of integrodifferential equations of the form

$$\mathrm{d}x/\mathrm{d}t = A(t)x + \int_\alpha^t K(t,s)x(s)\,\mathrm{d}s + f(t), \quad t \neq \theta_i,$$

$$\Delta x(\theta_i) = B_i x(\theta_i) + \sum_{j:\alpha<\theta_j\leqslant\theta_i} D_{ij}x(\theta_j) + \int_\alpha^{\theta_i} S_i(s)x(s)\,\mathrm{d}s + J_i. \tag{11}$$

Theorem 2. *Let conditions* (C2), (C3), *and* (C4) *be satisfied. Then for a given* $x_0 \in R^n$ *there exists a unique solution* $x(t) \in PAC[\alpha, \beta]$ *of* (11) *which satisfies* $x(\alpha) = x_0$.

Proof. It is easy to verify that the integrodifferential equation

$$x(t) = x_0 + \int_\alpha^t A(s)x(s)\,\mathrm{d}s + \int_\alpha^t \int_\alpha^\sigma K(\sigma,s)x(s)\,\mathrm{d}s\,\mathrm{d}\sigma + \sum_{i:\alpha<\theta_i<t} B_i x(\theta_i) + \int_\alpha^t f(s)\,\mathrm{d}s$$
$$+ \sum_{i:\alpha<\theta_i<t}\sum_{j:\alpha<\theta_j\leqslant\theta_i} D_{ij}x(\theta_i) + \sum_{i:\alpha<\theta_i<t}\int_\alpha^{\theta_i} M_i(s)x(s)\,\mathrm{d}s + \sum_{i:\alpha<\theta_i<t} J_i \tag{12}$$

is equivalent to (11) with $x(\alpha) = x_0$. Letting

$$\Psi(t,s) = A(s) + \int_s^t K(\sigma,s)\,\mathrm{d}\sigma + \sum_{j:s<\theta_j<t} M_j(s),$$

$$\Phi_i = B_i + \sum_{j:\theta_i\leqslant\theta_j<t} D_{ji},$$

$$F(t) = x_0 + \int_\alpha^t f(s)\,\mathrm{d}s,$$

and using Fubini's Theorem along with Lemma 2, it follows from (12) that

$$x(t) = \int_\alpha^t \Psi(t,s)x(s)\,\mathrm{d}s\,\mathrm{d}\sigma + \sum_{i:\alpha<\theta_i<t} \Phi_i x(\theta_i) + \sum_{i:\alpha<\theta_i<t} J_i + F(t). \tag{13}$$

In view of Theorem 1 we may conclude that (13) has a unique solution $x(t) \in \mathrm{PAC}[\alpha, \beta]$. This completes the proof. □

Next we consider the impulsive integrodifferential equation

$$\partial\lambda(t,s)/\partial s = -\lambda(t,s)A(s) - \int_s^t \lambda(t,\sigma)K(\sigma,s)\,\mathrm{d}\sigma - \sum_{j:s\leqslant\theta_j<t} \lambda(t,\theta_j)M_i(s), \quad s \neq \theta_i,\ t \in [\alpha,\beta],$$

$$\Delta\lambda(t,\theta_i) = -\lambda(t,\theta_i)B_i(I+B_i)^{-1} - \sum_{j:\theta_i\leqslant\theta_j<t} \lambda(t,\theta_j+)D_{ji}(I+B_i)^{-1}, \tag{14}$$

where $\lambda \in R^n$ is a row vector, A, K, D_{ij}, M_i and B_i are as before, and $\Delta\lambda(t, \theta_i) := \lambda(t, \theta_i+) - \lambda(t, \theta_i)$.

Theorem 3. *Let conditions* (C2)–(C5) *hold. Then for a given* $\lambda_0 \in R^n$ *system* (14) *has a unique solution* $\lambda(t, s)$ *such that* $\lambda(t, t) = \lambda_0$.

Proof. System (14) is equivalent to the integrodifferential equation

$$\lambda(t,s) = \lambda(t,t) + \int_s^t \left[\lambda(t,\sigma)A(\sigma) + \int_\sigma^t \lambda(t,v)K(v,\sigma)\,dv + \sum_{i:\sigma\leqslant\theta_i<t} \lambda(t,\theta_i)M_i(\sigma)\right] d\sigma$$
$$+ \sum_{i:s\leqslant\theta_i<t} \lambda(t,\theta_i)B_i(I+B_i)^{-1} + \sum_{i:s\leqslant\theta_i<t}\ \sum_{j:\theta_i\leqslant\theta_j<t} \lambda(t,\theta_j+)D_{ji}(I+B_i)^{-1}.$$

Using similar versions of Lemmas 2 and 3 and Fubini's Theorem we obtain that the previous equation is equivalent to

$$\lambda(t,s) = \lambda_0 + \int_s^t \lambda(t,\sigma)\left[A(\sigma) + \int_s^\sigma K(\sigma,v)\,dv\right] d\sigma + \sum_{i:s\leqslant\theta_i<t} \lambda(t,\theta_i)\Big[B_i(I+B_i)^{-1}$$
$$+ \int_s^{\theta_i} M_i(s)\,ds\Big] + \sum_{i:s\leqslant\theta_i<t} \lambda(t,\theta_i+) \sum_{j:s\leqslant\theta_j\leqslant\theta_i} D_{ij}(I+B_i)^{-1}. \tag{15}$$

Comparing (7) and (15) it is not difficult to see that the arguments developed in the proof of Theorem 1 can be used, and, therefore we can conclude that system (14) has a unique solution $\lambda(t, s)$ satisfying $\lambda(t, t) = \lambda_0$.

Now for every $i = 1, \ldots, n$, denote by $\lambda_i(t, s)$ the unique solution of (14) such that if $\Lambda(t, s) = \mathrm{col}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$, then $\Lambda(t, t) = I$. □

Theorem 4. *Let* $x(t) = x(t, \alpha, x_0)$ *be a solution of the Cauchy problem for* (11). *Then* $x(t)$ *has the representation*

$$x(t) = \Lambda(t,\alpha)x_0 + \int_\alpha^t \Lambda(t,s)f(s)\,ds + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_j+)J_i. \tag{16}$$

Proof. Let $x(t) = x(t, \alpha, x_0)$ be the solution of (11) and let $\phi(s) = \Lambda(t, s)x(s)$. Clearly,

$$\phi(t) - \phi(\alpha) = \int_\alpha^t \phi\prime(s)\,ds + \sum_{i:\alpha<\theta_i<t} \Delta\phi(\theta_i). \tag{17}$$

Since

$$\Delta\phi(\theta_i) = \Lambda(t,\theta_i+)x(\theta_i+) - \Lambda(t,\theta_i)x(\theta_i) = \Lambda(t,\theta_i)\Delta x(\theta_i) + \Delta\Lambda(t,\theta_i)x(\theta_i+),$$

we have

$$\sum_{i:\alpha<\theta_i<t} \Delta\phi(\theta_i) = \sum_{i:\alpha<\theta_i<t} [\Delta\Lambda(t,\theta_i)(I+B_i) + \Lambda(t,\theta_i)B_i]x(\theta_i) + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i)\int_\alpha^{\theta_i} M_i(s)x(s)\,ds$$
$$+ \sum_{i:\alpha<\theta_i<t}\ \sum_{j:\alpha<\theta_j\leqslant\theta_i} \Lambda(t,\theta_i+)D_{ij}x(\theta_j) + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i$$
$$= \sum_{i:\alpha<\theta_i<t}\left[\Delta\Lambda(t,\theta_i)(I+B_i) + \Lambda(t,\theta_i)B_i + \sum_{j:\theta_i\leqslant\theta_j<t} \Lambda(t,\theta_j+)D_{ji}\right]x(\theta_i)$$
$$+ \sum_{i:\alpha<\theta_i<t} \int_\alpha^{\theta_i} \Lambda(t,\theta_i)M_i(s)x(s)\,ds + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i. \tag{18}$$

On the other hand, by differentiating the relation $\phi(s) = \Lambda(t, s)x(s)$ we have

$$\phi'(s) = \partial\Lambda/\partial s x(s) + \Lambda(t,s)\left[A(s)x(s) + \int_\alpha^s K(s,v)x(v)\,dv + f(s)\right]$$

and therefore by Fubini's Theorem it follows that

$$\int_\alpha^t \phi'(s)\,ds = \int_\alpha^t \left\{[\partial\Lambda/\partial s + \Lambda(t,s)A(s)]x(s) + \Lambda(t,s)\int_\alpha^s K(s,v)x(v)\,dv\right\}ds + \int_\alpha^t \Lambda(t,s)f(s)\,ds)$$
$$= \int_\alpha^t \left[\partial\Lambda/\partial s + \Lambda(t,s)A(s) + \int_\alpha^t \Lambda(t,v)K(v,s)\,dv\right]x(s)\,ds + \int_\alpha^t \Lambda(t,s)f(s)\,ds. \tag{19}$$

Now (14), (17), (18) and (19) imply that

$$\phi(t) - \phi(\alpha) = \int_\alpha^t \left[\partial\Lambda/\partial s + \Lambda(t,s)A(s) + \int_\alpha^t \Lambda(t,v)K(v,s)\,dv\right]x(s)\,ds + \int_\alpha^t \Lambda(t,s)f(s)\,ds$$
$$+ \sum_{j:\theta_i\leqslant\theta_j<t} [\Delta\Lambda(t,\theta_i)(I+B_i) + \Lambda(t,\theta_i)B_i + \Lambda(t,\theta_i+)D_{ij}]x(\theta_i) + \int_\alpha^t \sum_{i:s\leqslant\theta_i<t} \Lambda(t,\theta_i)M_i(s)x(s)\,ds$$
$$+ \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i = \int_\alpha^t \Lambda(t,s)f(s)\,ds + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)J_i.$$

This completes the proof. □

4. Comparison method for integrodifferential impulsive system

To investigate problem $\Sigma_\mu(G_h)$ (Definition 2.1) we shall use a comparison method [2]. For this purpose, we need to construct an integrodifferential equation with fixed moments of impulse actions which is associated with (1). We propose

$$\begin{aligned} dy/dt &= A(t)y + \int_\alpha^t K(t,s)y(s)\,ds + C(t)u + f(t) + \mu g(t,y,u,\mu) + \sum_{i:\alpha<\theta_i<t} F_i(t,y,u,\mu), \quad t \neq \theta_i, \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{j:\alpha<\theta_j<\theta_i} D_{ij}y(\theta_j) + Q_i v_i + J_i + S_i(y,u,v_i,\mu), \end{aligned} \tag{20}$$

where F_i and S_i are some functionals to be determined. Without any loss of generality we may assume that $\tau_i \geqslant 0$ in G_H.

Definition 4.1. The systems (1) and (20) are said to have B-property in G_H, if for a fixed positive real number $h < H$ and a sufficiently small μ the following conditions are fulfilled:

Given any solution $x(t)$ of (1), $|x(t)| < h$, $t \in [\alpha, \beta]$, $x(t)$ meets every surface of discontinuity only once, then there is a solution $y(t)$ of (20), $|y(t)| < H$, such that $x(t) = y(t)$ for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$. Conversely, given any solution $y(t)$ of (20), $|y(t)| < h$, $t \in [\alpha, \beta]$, if a solution $x(t)$ of (1), $x(\alpha) = a$, $|x(t)| < H$, intersects every surface of discontinuity once, the condition $x(t) = y(t)$ holds for all $t \in [\alpha, \beta]$ except possibly at points $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$.

In this section, we show that it is possible to choose the functionals S_i and F_i in such a way that systems (1) and (20) have B-property in G_H.

Fix h and a positive real number $\mu_2 \leqslant \mu_1$ such that $\mu_2 m_3 d(\mu_2) < H - h$, where $d(\mu) = m_1 H(2 + (\beta-\alpha)) + m_2 + \mu m_3$.

Let $\phi(t)$ be a solution of (1) satisfying $\phi(\alpha) = a$ and $|\phi(t)| < h$ for $t \in [\alpha, \beta]$. Assume that $\phi(t)$ meets each surface of discontinuity only once. Denote by $\zeta_i, i = \overline{1,p}$, the points of discontinuity of $\phi(t)$. Using (4) and nonnegativeness of $\tau_i, i = \overline{1,p}$, one can see that $\alpha < \theta_1 \leqslant \zeta_1 < \theta_2 \leqslant \zeta_2 < \cdots \leqslant \zeta_{p-1} < \theta_p \leqslant \zeta_p < \beta$. We should emphasize first of all that a functional $F_i(t, y, u, \mu)$ vanishes if $t \notin [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$. Thus the right sides of the differential parts of Eqs. (1) and (20) are the same if t is not from $[\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$.

Fix k, $1 < k \leqslant p$, and suppose that F_i, S_i, $i = 1, 2, \ldots, k-1$, are determined so that the solution $\eta(t)$ of (20) with $\eta(\alpha) = a$ satisfies

(i) $|\eta(t)| < H$ for $t \in [\alpha, \theta_k]$,
(ii) $\eta(t) = \phi(t)$ for $t \in [\alpha, \theta_k] \setminus \bigcup_{i=1}^{k-1}(\theta_i, \zeta_i]$, and
(iii) $\int_\alpha^t [A(s)\eta(s) + \int_\alpha^s K(s,\sigma)\eta(\sigma)\,\mathrm{d}\sigma + C(s)u + f(s) + \sum_{i:\alpha<\theta_i<s} F_i(s,\eta,u,\mu)]\,\mathrm{d}s =$
$\int_\alpha^t [A(s)\phi(s) + \int_\alpha^s K(s,\sigma)\phi(\sigma)\,\mathrm{d}\sigma + C(s)u + f(s) + \mu g(s,\phi(s),u(s),\mu)]\,\mathrm{d}s$ for $t \in [\alpha, \theta_k]$.

Because of $\zeta_k \geqslant \theta_k$, both of solutions $\eta(t)$ and $\phi(t)$ are left continuous at θ_k. And, moreover, from $\zeta_k \geqslant \theta_k$ it implies that they are equal to each other on $(\zeta_{k-1}, \theta_k]$. Hence, the equality $\eta(\theta_k) = \phi(\theta_k)$ is correct.

Let

$$\eta(\theta_k+) = (I + B_k)\eta(\theta_k) + \sum_{j:\alpha<\theta_j\leqslant\theta_k} D_{kj}\eta(\theta_j) + \int_\alpha^{\theta_k} M_k(s)\eta(s)\,\mathrm{d}s + Q_k v_k + J_k + z, \tag{21}$$

where z is to be determined. Continue the solution $\eta(t)$ for all $t \in [\theta_k, \zeta_k]$ as the solution of the initial value problem

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = F(t), \quad \xi(\zeta_k) = \phi(\zeta_k+), \tag{22}$$

where

$$F(t) = A(t)\phi(t) + \int_\alpha^t K(t,\sigma)\phi(\sigma)\,\mathrm{d}\sigma + C(t)u + f(t) + \mu g(t,\phi(t),u(t),\mu).$$

It is clear that

$$\eta(t) = \phi(\zeta_k+) + \int_{\zeta_k}^t F(s)\,\mathrm{d}s \quad \text{for } t \in [\theta_k, \zeta_k].$$

Thus,

$$|\eta(t) - \phi(\zeta_k+)| \leqslant \max_{\theta_k\leqslant t\leqslant\zeta_k} \left| \int_{\theta_k}^t F(s)\,\mathrm{d}s \right| \leqslant \mu_3 m_3 \mathrm{d}(\mu_3) < H - h,$$

and hence $|\eta(t)| < H$ for $t \in [\theta_k, \zeta_k]$. Moreover, letting $S_k(\eta, u, v_k, \mu) = z$, we see in view of (21) and the fact that

$$\eta(\theta_k+) = \phi(\zeta_k+) + \int_{\zeta_k}^{\theta_k} F(s)\,\mathrm{d}s,$$

that

$$S_k(\eta,u,v_k,\mu) = \phi(\zeta_k+) + \int_{\zeta_k}^{\theta_k} F(s)\,\mathrm{d}s - (I + B_k)\eta(\theta_k) - \sum_{j:\alpha<\theta_j\leqslant\theta_k} D_{kj}\eta(\theta_j) - Q_k v_k - J_k.$$

It is also possible to rewrite S_k as

$$S_k(\eta, u, v_k, \mu) = B_k[(\phi(\zeta_k) - \eta(\theta_k)] + \sum_{j:\alpha<\zeta_j\leqslant\zeta_k} D_{kj}[\phi(\zeta_j) - \eta(\theta_j)] + \sum_{j:\alpha<\zeta_j<\zeta_k} \int_{\theta_j}^{\zeta_j} M_k(s)(\phi(s) - \eta(s))\,\mathrm{d}s + \int_{\theta_k}^{\zeta_k} M_k(s)\phi(s)\,\mathrm{d}s + \mu W_k(\phi(\zeta_k), v_k, \mu). \tag{23}$$

Finally, we let

$$F_k(t, \eta, u, \mu) = \begin{cases} A(t)[\phi(t) - \eta(t)] + \int_\alpha^t K(t,s)[\phi(s) - \eta(s)]\,\mathrm{d}s + \mu g(t, \phi(t), u(t), \mu), & t \in [\theta_k, \zeta_k], \\ 0, & \text{otherwise}. \end{cases} \tag{24}$$

Lemma 4. *Systems (1) and (20) have B-property in G_H.*

Proof. Let $\phi(t)$ be the solution of Eq. (1) which has been described above in the section. Let us assume that $\eta(t)$ is a solution of (20) and satisfies (i) and (ii) for a fixed $k, 1 \leqslant k < p$. Now, let us consider an interval $[\theta_k, \theta_{k+1}]$. It is clear that

$$\eta(\zeta_k+) = (I + B_k)\eta(\theta_k) + \sum_{j:\alpha<\theta_j\leqslant\theta_k} D_{kj}\eta(\theta_j) + Q_k v_k + J_k + S_k(\eta, u, v_k, \mu) + \int_{\theta_k}^{\zeta_k} \left[A(s)\eta(s) + \int_\alpha^s K(s,\sigma)\eta(\sigma)\,\mathrm{d}\sigma + C(s)u + f(s) + \sum_{i:\alpha<\theta_i<s} F_i(s, \eta, u, \mu) \right] \mathrm{d}s.$$

In view of (i)–(iii) and the definition of S_k, the above formula implies that $\eta(\zeta_k+) = \phi(\zeta_k+)$. It follows that $\phi(t) = \eta(t)$ for $t \in (\zeta_k, \theta_{k+1}]$ because of the common initial data $\phi(\zeta_k+)$. Since k is arbitrary, $\phi(t) = \eta(t)$ for all $t \in [\alpha, \beta] \setminus \bigcup_{i=1}^p [\theta_k, \zeta_k)$.

Furthermore, let $\eta(t)$ be a solution of (20) such that $|\eta(t)| < h$ for all $t \in [\alpha, \beta]$ and let $\phi(t)$ be a solution of (1), satisfying $\phi(\alpha) = \eta(\alpha)$. Let ζ_i be the points of discontinuity of $\phi(t)$. Assume that for a fixed k, $1 \leqslant k < p$, conditions (i), (ii) hold and $\phi(t)$ intersects the surface of discontinuity $t = \theta_i + \mu\tau_i(x, \mu), i = \overline{1, k-1}$, at $t = \zeta_i$ only once.

Obviously, we may extend $\phi(t)$ as a solution of (1) until it intersects the surface $t = \theta_k + \mu\tau_k(x, \mu)$. Denote this intersection point by ζ_k. By our assumption it is unique.

In view of (20) we see that $\eta(t) = \phi(t)$ on each section $[\zeta_k, \theta_{k+1}]$ as well. Continuing this process we obtain $\phi(t)$ defined on $[\alpha, \beta]$ and $\phi(t) = \eta(t)$ for all $t \in [\alpha, \beta] \setminus \bigcup_{i=1}^p [\theta_k, \zeta_k)$. This completes the proof. □

5. Controllability of impulsive integrodifferential equations

We first consider the controllability problem for (20), (2). This problem is denoted by $\gamma_\mu(G_H)$. Define

$$\Psi(t) = \int_\alpha^t E(s)E^T(s)\,\mathrm{d}s + \sum_{i:\alpha<\theta_i<t} P_i P_i^T, \tag{25}$$

where $E(t) = \Lambda(\beta, t)C(t)$ and $P_i = \Lambda(\beta, \theta_i+)Q_i$.

Lemma 5. *Let* (C1)–(C5) *be satisfied. If $\Psi(\beta)$ is nonsingular, $\gamma_\mu(G_H)$ is a solvable problem.*

Proof. We will prove that $\gamma_\mu(G_H)$ is solvable with a control $\{u, v\}$ of the following form:

$$u = E^T(t)c + \hat{u}(t), \quad t \in [\alpha, \beta], \qquad v_i = P_i^T c + \hat{v}_i, \quad i = 1, 2, \ldots, p, \tag{26}$$

where $c \in R^n$ is a constant vector, and $\{\hat{u}, \hat{v}\} \in \Pi^m[\alpha, \beta]$ is orthogonal to all columns of $[E^T, P_i^T]$. It is clear that solving problem $\gamma_\mu(G_H)$ is equivalent to solving

$$\begin{aligned} y(t) = {} & \Lambda(t,\alpha)a + \int_\alpha^t \Lambda(t,s)\left[C(s)u(s) + f(s) + \sum_{i:\alpha<\theta_i<s} F_i(s,y,u,\mu)\right] \mathrm{d}s \\ & + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)[Q_i v_i + J_i + S_i(y,u,v_i,\mu)], \quad y(\beta) = b. \end{aligned} \tag{27}$$

Substituting (26) into (27) we have

$$\begin{aligned} c = {} & \Psi^{-1}(\beta)\left\{ b - \Lambda(\beta,\alpha)a - \int_\alpha^\beta \Lambda(\beta,s)\left[f(s) + \sum_{i:\alpha<\theta_i<s} F_i(s,y,u,\mu)\right] \mathrm{d}s \right. \\ & \left. - \sum_{i=1}^p \Lambda(\beta,\theta_i+)[J_i + S_i(y,u,v_i,\mu)] \right\}. \end{aligned} \tag{28}$$

Let us set

$$\begin{aligned} & K = b - \Lambda(\beta,\alpha)a - \int_\alpha^\beta \Lambda(\beta,s)f(s)\,\mathrm{d}s - \sum_{i=1}^p \Lambda(t,\theta_i+)J_i, \\ & u_0(t) = E^T(t)\Psi(\beta)^{-1}K + \hat{u}(t), \quad v_i^0 = P_i^T\Psi(\beta)^{-1}K + \hat{v}_i, \\ & y_0(t) = \Lambda(t,\alpha)a + \int_\alpha^t \Lambda(t,s)[C(s)u_0(s) + f(s)]\,\mathrm{d}s + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)[Q_i v_i^0 + J_i], \\ & \varphi = (y(t), u(t), \{v_i\}_{i=1}^p), \quad \varphi_0 = (y_0(t), u_0(t), \{v_i^0\}_{i=1}^p), \\ & \kappa(t,\varphi,\mu) = \mu^{-1}\int_\alpha^t \Lambda(t,s) \sum_{i:\alpha<\theta_i<s} F_i(s,y,u,\mu)\,\mathrm{d}s, \\ & \psi(t,\varphi,\mu) = \mu^{-1} \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)S_i(y,u,v_i,\mu). \end{aligned} \tag{29}$$

Substituting in (27) expressions from (26) and (28), one can obtain that

$$\begin{aligned} y(t) = {} & \Lambda(t,\alpha)a + \int_\alpha^t \Lambda(t,s)\left\{ C(s)\left[E^T(s)\Psi^{-1}(\beta)\left[b - \Lambda(\beta,\alpha)a \right.\right.\right. \\ & \left.\left.\left. - \int_\alpha^\beta \Lambda(\beta,r)\left[f(r) + \sum_{i:\alpha<\theta_i<r} F_i(r,y,u,\mu)\right]\mathrm{d}r\right] + \hat{u}(s)\right] + f(s) + \sum_{i:\alpha<\theta_i<s} F_i(s,y,u,\mu) \right\} \mathrm{d}s \\ & + \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)\left\{ Q_i\left[P_i^T\Psi^{-1}(\beta)\left[b - \Lambda(\beta,\alpha)a - \int_\alpha^\beta \Lambda(\beta,r)[f(r) \right.\right.\right. \\ & \left.\left.\left. + \sum_{i:\alpha<\theta_i<r} F_i(r,y,u,\mu)]\,\mathrm{d}r\right] + \hat{v}_i\right] + J_i + S_i(y,u,v_i,\mu) \right\}. \end{aligned} \tag{30}$$

Then, using (29), we can rewrite

$$\begin{aligned} y(t) = {} & y_0(t) - \mu\int_\alpha^t \Lambda(t,s)C(s)E^T(s)\kappa(\beta,\varphi,\mu)\,\mathrm{d}s \\ & - \mu\sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)Q_iP_i^T\psi(\beta,\varphi,\mu) - \mu\kappa(t,\varphi,\mu) - \mu\psi(t,\varphi,\mu). \end{aligned} \tag{31}$$

Similarly one can write that

$$u(t) = u_0(t) - \mu E(t)^T \Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)], \tag{32}$$

and

$$v_i = v_0^i - \mu P_i^T \Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)]. \tag{33}$$

From (31)–(33) it follows that

$$\varphi = \varphi_0 + \mu\mathscr{P}(\varphi,\mu), \tag{34}$$

where $\mathscr{P} = (\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}^i)$,

$$\mathscr{P}_1(t,\varphi) = -\int_\alpha^t \Lambda(t,s)C(s)E^T(s)\kappa(\beta,\varphi,\mu)\,\mathrm{d}s - \sum_{i:\alpha<\theta_i<t} \Lambda(t,\theta_i+)Q_iP_i^T\psi(\beta,\varphi,\mu) - \kappa(t,\varphi,\mu) - \psi(t,\varphi,\mu),$$

$$\mathscr{P}_2(t,\varphi) = -E(t)^T\Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)],$$

$$\mathscr{P}^i(\varphi) = -P_i^T\Psi^{-1}(\beta)[\kappa(\beta,\varphi,\mu) + \psi(\beta,\varphi,\mu)].$$

Consider a set $\mathscr{B}$ of functions φ of the form $\varphi = (y(t), u(t), \{v_i\}_{i=p}^p)$ with norm

$$\|\varphi\| = \sup_t |y(t)| + \sup_t |u(t)| + \max_i |v_i|.$$

where $y(t) \in \mathrm{PAC}[\alpha, \beta]$ and $\{u, v\} \in \Pi^m[\alpha, \beta]$. We may assume that the real number h, fixed in the previous section, also satisfies $\|\varphi_0\| < h$ and that there is a positive real number $\mu_3 \leqslant \mu_2$ such that

$$\mu_3 \max_{\substack{\|\varphi\|\leqslant h \\ 0<\mu\leqslant\mu_1}} \|\mathscr{P}(\varphi,\mu)\| < h - \|\varphi_0\|.$$

Let $\mathscr{X} = \{\varphi : \varphi \in \mathscr{B}, \|\varphi\| \leqslant h\}$. It is easy to see that if $\mu \leqslant \mu_3$ then the operator $\phi_0 + \mu\mathscr{P}(\varphi,\mu)$ maps $\mathscr{X}$ into itself. We will show that if μ is sufficiently small then $\mu\mathscr{P}$ becomes a contraction mapping on $\mathscr{X}$. We first show that operators κ and ψ satisfy a Lipschitz condition with respect to φ. For this purpose let $\varphi_j = (\eta_j, u_j, \{v_i^{(j)}\}_{i=1}^p) \in \mathscr{X}$ for $j = 1, 2$. Fix k, $1 < k \leqslant p$, and let ϕ_j, $j = 1, 2$, be functions defined on $[\alpha, \theta_k]$ such that $\phi_j(t) = \eta_j(t)$ for $t \in [\alpha, \theta_k] \setminus \bigcup_{i=1}^{k-1}(\theta_i, \zeta_i^{(j)}]$, where $\zeta_i^{(j)}$ is point of discontinuity of ϕ_j, and for all $t \in [\alpha, \theta_k]$,

$$\int_\alpha^t \left[A(s)\phi_j(s) + \int_\alpha^s K(s,\sigma)\phi_j(\sigma)\,\mathrm{d}\sigma + \mu g(s,\phi_j(s),u_j(s),\mu)\right]\mathrm{d}s$$
$$= \int_\alpha^t \left[A(s)\eta_j(s) + \int_\alpha^s K(s,\sigma)\eta_j(\sigma)\,\mathrm{d}\sigma + \sum_{i:\alpha<\theta_i<s} F_i(s,\eta_j,u_j,\mu)\right]\mathrm{d}s. \tag{35}$$

Next we shall apply the method of mathematical induction. Assume that

$$|\phi_1(t) - \phi_2(t)| \leqslant l_1(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } t \in [\alpha,\theta_k]\Big\backslash \bigcup_{i=1}^{k-1}(\zeta_i^{(1)}, \zeta_i^{(2)}] \tag{36}$$

and

$$\zeta_i^{(2)} - \zeta_i^{(1)} \leqslant \mu l_2(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } i = 1,2,\ldots,k-1, \tag{37}$$

where $l_1(\mu)$ and $l_2(\mu)$ are some bounded functions, and without any loss of generality, $\zeta_i^{(2)} \geqslant \zeta_i^{(1)}$ for $i = 1, 2, \ldots, p$. Now we continue $\phi_j(t)$ as a solution of the following equation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = A(t)\phi(t) + \int_\alpha^t K(t,s)\phi(s)\,\mathrm{d}s + C(t)u_j(t) + f(t) + \mu g(t,\phi,u_j,\mu),$$

with the initial condition $\phi(\theta_k) = \eta_j(\theta_k)$ until it meets a surface $t = \theta_k + \mu\tau_k(x, \mu)$ when $t = \zeta_k^{(j)}$. It is clear that for $t \in [\theta_k, \zeta_k^{(j)}]$,

$$\phi_j(t) = \eta_j(\theta_k) + \int_{\theta_k}^{t} \left[A(s)\phi_j(s) + \int_{\alpha}^{s} K(t,\sigma)\phi_j(\sigma)\,d\sigma + C(t)u_j(t) + f(t) + \mu g(t, \phi_j(s), u_j(s), \mu)\right] ds. \tag{38}$$

Assume that $|\phi_j(t)| < H$ for all $t \in [\alpha, \theta_k]$. We claim that this inequality is also valid for all $t \in [\theta_k, \zeta_k^{(j)}]$. For if it is not true for the first time at some point $t^* \in (\theta_k, \zeta_k^{(j)}]$, then using (38) we have $|\phi_j(t)| \leqslant h + \mu_3 m_3 d(\mu_2) < H$ for all $t \in [\theta_k, t^*]$ and, in particular, $|\phi_j(t^*)| < H$. This contradiction verifies our claim. Now, using (36)–(38), we obtain

$$|\phi_1(t) - \phi_2(t)| \leqslant l_3(\mu)\|\varphi_1 - \varphi_2\| \quad \text{for } t \in [\theta_k, \zeta_k^{(1)}], \tag{39}$$

where $l_3(\mu) = (1 + l_1(\mu)(\beta - \alpha) + 2\mu p l_2(\mu)H + \mu m_3(m_1 + \mu L)(1-\mu m_3(m_1(1 + \mu m_3) + \mu L))^{-1}$. On the other hand, since

$$\zeta_k^{(2)} - \zeta_k^{(1)} = \mu\tau_k(\phi_2(\zeta_k^{(2)}), \mu) - \mu\tau_k(\phi_1(\zeta_k^{(1)}), \mu) \leqslant \mu L|\phi_2(\zeta_k^{(2)}) - \phi_2(\zeta_k^{(1)}) - \phi_2(\zeta_k^{(1)}) + \phi_1(\zeta_k^{(1)})|,$$

we have

$$\zeta_k^{(2)} - \zeta_k^{(1)} \leqslant \mu l_4(\mu)\|\varphi_1 - \varphi_2\|, \tag{40}$$

where $l_4(\mu) = \mu L l_3(\mu)(1 - \mu L d(\mu))^{-1}$.

Now, we may consider S_k given by (23) and write

$$\begin{aligned}
&S_k(\eta_1, u_1, v_k^{(1)}, \mu) - S_k(\eta_2, u_2, v_k^{(2)}, \mu) \\
&\quad = B_k(\phi_1(\zeta_k^{(1)}) - \eta_1(\theta_k) - \phi_2(\zeta_k^{(1)}) + \eta_2(\theta_k)) + B_k(\phi_2(\zeta_k^{(1)})(-\phi_2(\zeta_k^{(2)})) \\
&\qquad + \sum_{j:\alpha<\zeta_j\leqslant\zeta_k} [D_{kj}(\phi_1(\zeta_j^{(1)})) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(1)}) + \eta_2(\theta_j)] + [D_{kj}(\phi_2(\zeta_j^{(1)}) - \phi_2(\zeta_j^{(2)}))] \\
&\qquad + \sum_{j:\alpha<\zeta_j<\zeta_k} \int_{\theta_j}^{\zeta_j^{(1)}} M_k(s)[\phi_1(s) - \phi_2(s) + (\eta_2(s) - \eta_1(s))]\,ds \\
&\qquad - \sum_{j:\alpha<\zeta_j<\zeta_k} \int_{\zeta_j^{(1)}}^{\zeta_j^{(2)}} M_k(s)[(\phi_2(s) - \eta_2(s))]\,ds + \int_{\theta_k}^{\zeta_k^{(1)}} M_k(s)(\phi_2(s) - \phi_1(s))\,ds \\
&\qquad + \int_{\zeta_k^{(1)}}^{\zeta_k^{(2)}} M_k(s)\phi_2(s)\,ds + \mu(W_k(\phi_1(\zeta_k^{(1)}, v_k^{(1)}, \mu)(-W_k(\phi_2(\zeta_k^{(2)}, v_k^{(2)}, \mu)))).
\end{aligned} \tag{41}$$

In view of (38),

$$\begin{aligned}
&\phi_1(\zeta_j^{(1)}) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(2)}) + \eta_2(\theta_j) \\
&\quad = \int_{\theta_k}^{\zeta_k^{(1)}} \left\{A(s)[\phi_1(s) - \phi_2(s)] + \int_{\alpha}^{s} K(s,\sigma)[\phi_1(\sigma) - \phi_2(\sigma)]\,d\sigma + C(s)[u_1(s) - u_2(s)]\right. \\
&\qquad \left. + \mu[g(s, \phi_1(s), u_1(s), \mu) - g(s, \phi_2(s), u_2(s), \mu)]\right\} ds \\
&\qquad + \int_{\zeta_k^{(1)}}^{\zeta_k^{(2)}} \left\{A(t)\phi_2(t) + \int_{\alpha}^{t} K(t,s)\phi_2(s)\,ds + C(t)u_2(t) + f(t) + \mu g(t, \phi_2(t), u_2(t), \mu)\right\} dt
\end{aligned}$$

and, therefore, by using (39) and (40) we have

$$|\phi_1(\zeta_j^{(1)}) - \eta_1(\theta_j) - \phi_2(\zeta_j^{(2)}) + \eta_2(\theta_j)| \leqslant \mu l_5(\mu)\|\varphi_1 - \varphi_2\|, \tag{42}$$

where $l_5(\mu) = m_3[m_1(l_3(\mu) + l_1(\mu)(\beta-\alpha) + \mu l_2(\mu)p + 1) + \mu L(1 + l_3(\mu))] + l_4(\mu)d(\mu)$, and $d(\mu) = m_1 H (2 + (\beta - \alpha)) + m_2 + \mu m_3$.

It follows from the last inequality and (41) that

$$|S_k(\eta_1, u_1, v_k^{(1)}, \mu) - S_k(\eta_2, u_2, v_k^{(2)}, \mu)| \leqslant \mu l^{(1)}(\mu)\|\varphi_1 - \varphi_2\|,$$

where $l^{(1)}(\mu) = m_1[(p + 1)l_5(\mu) + l_4(\mu)(d(\mu) + H) + m_3(l_3(\mu)(p + 1) + p)] + Hl_4(\mu)(2p + 1)m_1 + L(2 + \mu l_4(\mu)d(\mu))$ is a bounded function. Since there are only finitely many S_j, it is clear that for some constant $L_1(\mu)$,

$$|S_i(\eta_1, u_1, v_i^{(1)}, \mu) - S_i(\eta_2, u_2, v_i^{(2)}, \mu)| \leqslant \mu L^{(1)}(\mu)\|\varphi_1 - \varphi_2\| \tag{43}$$

for $i = 1, 2, \ldots, p$.

Similarly, by using the representation of F_i given in (24) we can show that there is a bounded function $L^{(2)}(\mu)$ such that

$$\left| \sum_{i:\alpha<\theta_i<t} F_i(t, \eta_2, u_2, \mu) - \sum_{i:\alpha<\theta_i<t} F_i(t, \eta_1, u_1, \mu) \right| \leqslant \mu L^{(2)}(\mu)\|\varphi_1 - \varphi_2\|, \tag{44}$$

uniformly for all $t \in [\alpha, \beta]$.

Let $m_4 = \max_{t,s}|\Lambda(t, s)|$. From (43) and (44) it follows that

$$|\psi(t, \varphi_1, \mu) - \psi(t, \varphi_2, \mu)| \leqslant pm_4 L^{(1)}(\mu)\|\varphi_1 - \varphi_2\|$$

and

$$|\kappa(t, \varphi_1, \mu) - \kappa(t, \varphi_2, \mu)| \leqslant (\beta - \alpha)m_4 L^{(2)}(\mu)\|\varphi_1 - \varphi_2\|.$$

Thus, we have

$$|\kappa(t, \varphi_1, \mu) - \kappa(t, \varphi_2, \mu)| + |\psi(t, \varphi_1, \mu) - \psi(t, \varphi_2, \mu)| \leqslant \mu L(\mu)\|\varphi_1 - \varphi_2\|,$$

where $L(\mu) = (\beta - \alpha)L^{(2)}(\mu) + pm_4 L^{(1)}(\mu)$.

Finally, letting

$$m_5 = \max\left\{ \max_t \|\Psi(t)\Psi^{-1}(\beta)\|, \max_t \|E^T(t)\Psi^{-1}(\beta)\|, \max_i \|P_i^T\Psi^{-1}(\beta)\| \right\},$$

we can verify that

$$\|\mathscr{P}(\varphi_1, \mu) - \mathscr{P}(\varphi_2, \mu)\| \leqslant 2L(\mu)(1 + 6m_5)\|\varphi_1 - \varphi_2\|.$$

Thus, if we choose $\mu^0 \leqslant \mu_3$ sufficiently small so that $2\mu L(\mu)(1 + 6m_5) < 1$ for all $0 < \mu < \mu^0$, then the operator $\mu\mathscr{P}$ becomes contractive. So there is a fixed point φ^0 of the operator $\phi_0 + \mu\mathscr{P}(\varphi, \mu)$. It is easy to verify that φ^0 is a solution of problem $\gamma_\mu(G_H)$, and therefore the proof is complete. □

We shall need the following condition:

(C6) The inequality

$$\tau_i(y_0(\theta_i), 0) > \tau_i(y_0(\theta_i+), 0) \tag{45}$$

is valid for all a, b, $|a| \leqslant h$, $|b| \leqslant h$, and $i = 1, 2, \ldots, p$.

Theorem 5. *Let* (C1)–(C6) *be satisfied. If* $\Psi(\beta)$ *is nonsingular, then* $\Sigma_\mu(G_h)$ *is solvable.*

Proof. Using Lemma 5, we can check that there is a control $\{\hat{u}, \hat{v}\}$ for which system (20) admits a solution $y(t)$ such that $y(\alpha) = a$ and $y(\beta) = b$.

Let us assume that a solution $x(t)$, $x(\alpha) = a$, of (1) has no intersection with the surfaces $t = \theta_i + \mu\tau_i(x, \mu)$, $i = 1, 2, \ldots, k-1$, $1 < k < p$. In this case, according to Lemma 4, we have that $x(t) = y(t)$ for all $t \in [\alpha, \zeta_k]$ except possibly for $t \in [\theta_i, \zeta_i]$, $i = \overline{1, k}$.

As

$$\zeta_k - \theta_k = \mu\tau_k(x(\zeta_k), \mu)$$

and

$$x(t) = y(\theta_k) + \int_{\theta_k}^{t}\left[A(s)x(s) + \int_{\alpha}^{s} K(s,\sigma)x(\sigma)\,\mathrm{d}\sigma + C(s)u(s) + f(s) + \mu g(x(s), u(s), \mu)\right]\mathrm{d}s, \quad t \in [\theta_k, \zeta_k],$$

by using (34), by Lipschitz condition on τ_k and by comparing the expressions for $\Delta y_0(\theta_k)$ and $\Delta x(\zeta_k)$, we have

$$\|x(\zeta_k+) - y_0(\theta_k+)\| \leqslant \mu\left[m_3(d(\mu) + 2pm_1H) + \max_{\substack{\|\phi\|\leqslant h\\ 0<\mu\leqslant\mu_1}} \|\mathscr{P}(\varphi, \mu)\|\right].$$

From the last inequality and (45) it follows that if μ is sufficiently small, then

$$\tau_k(x(\zeta_k), \mu) \geqslant \tau_k(x(\zeta_k+), \mu). \tag{46}$$

Substituting expressions for c, K, u_0, y_0 from (28) and (29) in (27) one can obtain continuous dependence of $y_0(t)$, regarded as a composed function, on a and b in *sup*–norm on $[\alpha, \beta]$. Hence, as k is arbitrary, and p is a finite number, there exists a positive real number μ_0, $\mu_0 < \mu^0$, such that relations $\mu L d(\mu) < 1$ and (46) are valid if $\mu < \mu_0$. Now, repeating the arguments from the proof of Lemma 5 on page 22 in [35] and using Lemma 1 we see that the solution $x(t)$ meets every surface of discontinuity once.

But in this case according to Lemma 4 we have $x(t) = y(t)$ for all $t \in [\alpha, \beta] \setminus \bigcup_{i=1}^{p}[\theta_i, \zeta_i]$. It is also true that $x(\beta) = b$. Herewith, the theorem is proved. □

6. Optimal control of response

In this section, we are going to include the aspect of optimality into our reflections. Firstly, this research was presented in paper [4]. This will give us a *time-minimal* problem which we shall in physical, economical or, possibly, social terms interpret as ''resonance''. Those time-optimal problems which we introduced in Section 1, aim at a minimal time consumption, a nearest to presence time horizon of a process in order reach a terminal state. Besides of our optimal response problem, there are, in particular, problems of time-minimal heating or cooling of a rigid body, say, of a ball consisting of a homogeneous material (cf., e.g., [27,28,37,38]). In those problems, the process is subject to both constraints on thermal thrust tangential to the boundary and some heat equation (the control variable being the temperature at the boundary). This gives rise to a two-stage optimization problem with a generalized semi-infinite programming problem on the upper stage ([27,28,37]). In our problem of optimal response, however, the process will be governed by a linear impulsive integrodifferential equation. This will by Pontryagin's maximum (or, minimum) principle lead us to *linear* programming problems.

Concerning the system then to be controlled optimally, we consider a particular linear case of equation and impulse (1):

$$\begin{aligned} &\mathrm{d}x/\mathrm{d}t = A(t)x + C(t)u(t) + f(t), \quad t \neq \theta_i, \\ &\Delta x(\theta_i) = B_i x(\theta_i) + Q_i v_i + J_i, \quad i = 1, 2, \ldots, p, \end{aligned} \tag{47}$$

$$x(0) = a, \quad x(\beta) = b, \tag{48}$$

where A and C are defined for any time $t \geqslant 0$, B_i and Q_i are bounded sequences, and time $\beta > 0$ is arbitrary. Before considering the main problem of the section let us investigate controllability of the problem for a fixed positive number $\beta \in R^n$. We say that the control problem $\mathscr{A}(\beta)$ is solvable if, for any f, $\{J_i\} \in \Pi^m[0, \beta]$ and $a, b \in R^n$, there exists $\{u, v\} \in \Pi^m[0, \beta]$ for which the boundary-value problem (47), (48) has a solution. The validity of the following statement can be verified similarly to the proof of Theorem 19.2 from [35].

Lemma 6. *Let F, $\{V_i\} \in \Pi^m[0, \beta]$. Then the boundary-value problem*

$$\begin{aligned} &\mathrm{d}x/\mathrm{d}t = A(t)x + F(t), \quad t \neq \theta_i, \ i = 1, 2, \ldots, p, \\ &\Delta x(\theta_i) = B_i x(\theta_i) + V_i, \end{aligned} \tag{49}$$

$$x(0) = 0, \quad x(\beta) = 0, \tag{50}$$

is solvable if and only if, for any solution $y(t)$ of the system

$$\begin{aligned} &\mathrm{d}y/\mathrm{d}t = -A^T(t)y, \quad t \neq \theta_i, \ i = 1, 2, \ldots, p, \\ &\Delta y(\theta_i) = -(I + B_i^T)^{-1} B_i^T y(\theta_i), \end{aligned} \tag{51}$$

the following relation holds:

$$\langle \{F, V_i\}, \{y, y(\theta_i)\} \rangle = 0. \tag{52}$$

Let $Y(t)(y_1, y_2, \ldots, y_n)$, be a fundamental matrix of solutions of adjoint system (51).

Theorem 6. The control problem $\mathscr{A}(\beta)$ is solvable if and only if

$$\int_0^\beta Y^T(t)[C(t)u(t) + f(t)]\,\mathrm{d}t + \sum_{i:0<\theta_i<\beta} Y^T(\theta_i)[Q_i v_i + J_i] = Y^T(\beta)b - Y^T(0)a. \tag{53}$$

Proof. Let us change the variables $x = z + \kappa(t)$ in the boundary-value problem $\mathscr{A}(\beta)$. Here, κ is an arbitrary function continuous with its derivatives and satisfying the boundary condition, i.e., $\kappa(0) = a$, $\kappa(\beta) = b$, and the conditions $\kappa(\theta_i) = 0, i = \overline{1, p}$. Such a function can be easily constructed. For instance, one can take the Lagrange polynomial as κ. After the change of variables, we obtain the control problem $\mathscr{A}(\beta)$ in the form

$$\mathrm{d}z/\mathrm{d}t = A(t)z + C(t)u(t) + f(t) + [\kappa'(t) - A(t)\kappa(t)], \quad t \neq \theta_i,$$

$$\Delta z(\theta_i) = B_i z(\theta_i) + Q_i v_i + J_i, \quad i = 1, 2, \ldots, p, \tag{54}$$

$$z(0) = 0, \quad z(\beta) = 0. \tag{55}$$

By virtue of Lemma 6, for the solvability of this problem it is necessary and sufficient that the condition

$$\int_0^\beta Y^T(t)[C(t)u(t) + f(t)]\,\mathrm{d}t + \sum_{i:0<\theta_i<\beta} Y^T(\theta_i)[Q_i v_i + J_i] = \int_0^\beta Y^T(t)[\kappa'(t) - A(t)\kappa(t)]\,\mathrm{d}t, \tag{56}$$

where $\{u, v\} \in \Pi^m[0, \beta]$, be satisfied. Let y_k be the kth column of the matrix $Y(t)$. Integrating by parts, we obtain

$$\int_0^\beta (y_k, \kappa')\,\mathrm{d}t = (y_k(\beta), \kappa(\beta)) - (y_k(\alpha), \kappa(\alpha)) - \int_0^\beta (y_k', \kappa)\,\mathrm{d}t.$$

From this and (56), we conclude that the validity of relation (53) is the necessary and sufficient condition of solvability of the control problem $\mathscr{A}(\beta)$. The theorem is proved. □

Let us fix a positive number β and define the space-product $\Pi^m[0,\beta] = L_2^m[0,\beta] \times D^m[1]$ for which $\theta_i, i = \overline{1,p}$, are the points of discontinuity of functions from $L_2^m[0,\beta]$, which form an ordered sequence in the interval $(0, \beta)$.

We assume that the control $\{u, v\}$ can be chosen only from the set $\Delta \times \Delta' \subset \Pi^m[0, \beta]$ which is bounded in the norm $\|\cdot\|_{[0,\beta]}$. To solve the problem of fast response means to find by using a given element $\{f, J\}$ belonging to the space $\Pi^m[0, \beta]$ for any $\beta > 0$ the control $\{u, v\}$ that solves the problem in minimal time:

$$(\mathrm{OCR}) \quad \begin{cases} \text{minimize } \beta \\ \text{subject to (41) and (42).} \end{cases} \tag{57}$$

We say that a control $\{u, v\}$ with a vector $c = c_0$ in the domain $\Delta \times \Delta'$ satisfies the Pontryagin condition [34] if in the domain this control provides the maximum of the expression $c_0^T Y^T(t)C(t)u(t)$ for almost all $t \in [0, \beta]$, and the maximum of the expression $c_0^T Y^T(\theta_i)Q_i v_i, i = \overline{1,p}$.

Theorem 7. *Let a control $\{u, v\}$ solves the problem of control (47), (48) for the time $\beta > 0$, and let it satisfy the Pontryagin condition for some vector $c = c_0$ in the domain $\Delta \times \Delta'$. Suppose that the expression $c_0^T Y^T(t)[C(t)u(t) + f(t) + A(t)]b$ is positive for almost all $t \in [0, \beta]$ and the numbers $c_0^T Y^T(\theta_i)[Q_i v_i + J_i + B_i(I + B_i)^{-1})b], i = \overline{1,p}$, are positive. Then the control $\{u, v\}$ and trajectory $x(t)$ corresponding to it are optimal for (OCR), i.e., in the sense of fast response.*

Proof. The fundamental matrix of solutions, $Y(t)$, of the adjoint system (51) satisfies the equality [35]

$$Y(\beta) = Y(0) - \int_0^\beta A^T(t)Y(t)\,\mathrm{d}t + \sum_{i:0<\theta_i<\beta} (I + B_i^T)B_i^T Y(\theta_i).$$

Using the last expression from (53), carrying out the transposition, and assuming $Y(0) = I$, we get

$$\int_0^\beta Y^T(t)[C(t)u(t) + f(t) + A(t)b]\,\mathrm{d}t + \sum_{i:0<\theta_i<\beta} Y^T(\theta_i)[Q_i v_i + J_i + B_i(I + B_i)^{-1})b] = b - a. \tag{58}$$

Suppose the contrary holds, i.e., that there exists the control $\{\bar{u}, \bar{v}\}$ which transfers the point $x = a$ into position $x = b$ at the time $\tau < \beta$. Then,

$$\int_0^\tau c_0^T Y^T(t)[C(t)\bar{u}(t) + f(t) + A(t)b]\,\mathrm{d}t + \sum_{i:0<\theta_i<\tau} c_0^T Y^T(\theta_i)[Q_i\bar{v}_i + J_i + B_i(I + B_i)^{-1})b] = c_0^T(b - a).$$

Subtracting the last equality termwise from (54) multiplied by the vector c_0^T, we get

$$\int_0^\tau c_0^T Y^T(t)C(t)[u(t) - \bar{u}(t)]\,\mathrm{d}t + \int_\tau^\beta c_0^T Y^T(t)[C(t)u(t) + f(t) + A(t)b]\,\mathrm{d}t + \sum_{i:0<\theta_i<\tau} c_0^T Y^T(\theta_i)Q_i[v_i - \bar{v}_i]$$
$$+ \sum_{i:\tau<\theta_i<\beta} c_0^T Y^T(\theta_i)[Q_i v_i + J_i + B_i(I + B_i)^{-1})b] = 0.$$

By virtue of the conditions of the theorem, the first, the third and the fourth terms in the last equality are nonnegative, while the second one is positive. The resulting contradiction proves the theorem. □

Similarly to [34] one can check that the last theorem implies the validity of the following assertion.

Theorem 8. *Suppose that the conditions of Theorem 7 are satisfied, and $x(t)$ is the optimal trajectory connecting points $x(0) = a$ and $x(\beta) = b$. Then, any part of this trajectory which connects points $x(t_1)$ and $x(t_2)$, $0 \leqslant t_1 \leqslant t_2 \leqslant \beta$, is also an optimal trajectory.*

7. Conclusion

In this paper, we gave a contribution to the understanding of impulsive phenomena in various processes. For this emerging field of phenomena with its wide range of practical applications and methods, our pioneering paper combines control theory and optimization on the one hand, with quasilinear impulsive integrodifferential equations on the other hand. For the control problem, we presented results on existence, comparison methods and controllability, before we turned to optimal control of resonance. By this paper we want to recommend future research in more general nonlinear classes of impulsive optimal control problems from science, technology, economy and society. Among these problems, we also think of time-maximal ones on anticipation and prediction (see, e.g., [16]). We invite to deep interdisciplinary research using the theories of *impulsive* equations and (*dis*)continuous optimization.

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References

[1] M.U. Akhmet, J. Gebert, H. Öktem, S.W. Pickl, G.-W. Weber, An improved method for analytical modeling and anticipation of gene expression patterns, Journal of Computational Technologies (4) (2005) (cf. also www.iam.metu.edu.tr/research/preprint/6f.pdf).
[2] M.U. Akhmetov, N.A. Perestyuk, On comparison method for differential systems with pulse effect, Differential Equations 9 (26) (1990) 1079–1086.
[3] M.U. Akhmetov, N.A. Perestyuk, Periodic and almost periodic solutions of integro-differential equations with impulse actions, Matematicheskaya Fizika i Nelineinaya Mekhanika 42 (1987) 5–9.
[4] M.U. Akhmetov, N.A. Perestyuk, M.A. Tleubergenova, The control of linear impulsive systems, Ukrainskii Matematicheskii Zhurnal 47 (1995) 307–314.
[5] M. Akhmetov, R. Sejilova, The control of the boundary value problem for linear impulsive integrodifferential systems, Journal of Mathematical Analysis and Applications 236 (1999) 312–326.
[6] M.U. Akhmetov, A. Zafer, The controllability of boundary-value problems for quasilinear impulsive systems, Nonlinear Analysis 34 (1998) 1055–1065.
[7] L. Anokhin, L. Berezansky, E. Braverman, Stability of linear delay impulsive differential equations, Dynamic Systems and Applications 4 (1995) 173–188.
[8] A.A. Aslanyan, Conditions for optimum in problems of control of impulsive systems, Akademiya Nauk Ukrainskoi SSR Doklady, Seriya A 11 (1982) 3–6.
[9] K. Balachandran, D. Lalitha, Controllability of nonlinear Volterra integro-differential systems with prescribed controls, Journal of Applied Mathematics and Stochastic Analysis 5 (1992) 139–146.
[10] B. Ballobás, A.G. Thomason, Threshold functions, Combinatorica 7 (1987) 35–38.
[11] R. Bellman, Mathematical Methods in Medicine, World Scientific, Singapore, 1983.
[12] A. Bensoussan, J.L. Lions, Nouvelle formulation de problèmes de contrôle impulsionnel et applications, Comptes Rendus de l Academie des Sciences Paris 276 (1973) 1182–1189.
[13] C. Corduneanu, Sur une equation integrale de la théorie du réglage automatique, Comptes Rendus de l Academie des Sciences Paris 256 (1967) 3564–3567.

[14] A.L. Dontchev, W.W. Hager, A.B. Poore, B. Yang, Optimality, stability, and convergence in nonlinear control, Applied Mathematics and Optimization 31 (1995) 297–326.
[15] P. Erdös, A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hungar. Acad. Sci 5 (1960) 17–61.
[16] T. Ergenc, S.W. Pickl, N. Radde, G.-W. Weber, On the topology of generalized semi-infinite optimization and anticipatory systems, International Journal Computing Anticipatory Systems 14–16 (2004).
[17] J. Gebert, M. Lätsch, S.W. Pickl, G.-W. Weber, R. Wünschiers, An algorithm to analyze stability of gene-expression patterns, Discrete Mathematics and Data Mining of Discrete Applied Mathematics (special issue), E. Boros, P.L. Hammer, A. Kogan (guest editors), in press.
[18] J. Gebert, H. Öktem, S.W. Pickl, N. Radde, G.-W. Weber, F.B. Yilmaz, Inference of gene expression patterns by using a hybrid system formulation—an algorithmic approach to local state transition matrices, in: Proceedings of IIAS (International Institute for Advanced Studies), 16th International Symposium for Anticipatory and Predictive Systems, Baden Baden, July–August 2004 (cf. also www.iam.metu.edu.tr/research/preprint/20f.pdf).
[19] J. Gebert, N. Radde, Modelling gene regulatory networks with piecewise linear differential equations, preprint, Center of Applied Computer Science, University of Cologne, and talks held at EURO Summer Institute "Optimization and Data Mining", Ankara, Turkey, July 9–25, 2004. Available from: <http://www.iam.metu.edu.tr/research/preprint/8f.pdf>.
[20] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
[21] A. Halanay, D. Wexler, Qualitative Theory of Impulse Systems, Editura Academiei Republici Socialiste Romania, Bucuresti, 1968.
[22] M. Hestenes, Calculus of Variations and Optimal Control Theory, John Wiley, New York, 1966.
[23] K.H. Hofmann, Analysis I, An Introduction to Mathematics via Analysis, Berliner Studienreihen zur Mathematik 8, Heldermann Verlag, Lemgo, 2000.
[24] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
[25] A. Kaplan, R. Tichatschke, On a class of terminal variational problems, in: J. Guddat, H.Th. Jongen, F. Nozicka, G. Still, F. Twilt (Eds.), Parametric Optimization and Related Topics IV, Peter Lang, Frankfurt, Berlin, New York, 1996, pp. 185–199.
[26] A. Kaplan, R. Tichatschke, On the numerical treatment of a class of semi-infinite terminal problems, Optimization 41 (1997) 1–36.
[27] W. Krabs, On time-minimal heating or cooling of a ball, International Series of Numerical Mathematics 81, Birkhäuser, Basel, 1987, pp. 121–131.
[28] E. Kropat, S. Pickl, A. Rößler, G.-W. Weber, A new algorithm from semi-infinite optimization for a problem of time-minimal control, Journal of Computational Technologies 5 (4) (2000) 67–81.
[29] E. Kropat, S. Pickl, A. Rößler, G.-W. Weber, On theoretical and practical relations between discrete optimization and nonlinear optimization, Colloquy Optimization—Structure and Stability of Dynamical Systems of Journal of Computational Technologies 7 (2002) 27–62 (special issue).
[30] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, New York, 1989.
[31] E.B. Lee, L. Markus, Foundations of Optimal Control Theory, John Wiley, New York, 1967.
[32] K. Malanowski, Regularity of solutions in stability analysis of optimization and optimal control problems, Control and Cybernetics 23 (1994) 61–86.
[33] K. Malanowski, H. Maurer, Sensitivity analysis for parametric control problems with control-state constraints, Computational Optimization Applications 5 (1996) 253–283.
[34] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, John Wiley, New York, 1962.
[35] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[36] S. Sivasundram, Controllability and observability of nonlinear integro-differential systems, Proceedings of Dynamic Systems and Applications 2 (1996) 507–516.
[37] G.-W. Weber, Generalized Semi-infinite Optimization and Related Topics, Heldermann Verlag, Lemgo, 2003.
[38] G.-W. Weber, Generalized semi-infinite optimization: Theory and applications in optimal control and discrete optimization, Journal of Statistics & Management Systems 5 (1–3) (2002) 359–388.
[39] G.-W. Weber, New projects at the Institute of Applied Mathematics in METU, talk held at Workshop on advances in continuous optimization, Istanbul, Turkey, July 4–5, 2003.