# Periodic solutions of the hybrid system with small parameter

M. U. Akhmet[1,2,*], C. Büyükadalı[1], T. Ergenç[1,2]

[1] Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

[2] Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey

**Abstract**

In this paper we investigate the existence and stability of the periodic solutions of a quasilinear differential equation with piecewise constant argument. The continuous and differentiable dependence of the solutions on the parameter and the initial value is considered. A new Gronwall-Bellman type lemma is proved. Appropriate examples are constructed.

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## 1 Introduction and Preliminaries

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers and integers, respectively.

The main purpose of this paper is to apply the method of the small parameter to the following quasilinear system

$$x'(t) = A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu), \tag{1}$$

where $t \in \mathbb{R},\ x \in \mathbb{R}^n$ and $\mu$ is a small parameter belonging to an interval $J \subset \mathbb{R}$ with $0 \in J$; $f(t),\ g(t, x, y, \mu)$ are $n$ dimensional vectors, $A(t)$ is an $n \times n$ matrix

*Corresponding author, e-mail: marat@metu.edu.tr

for $n \in \mathbb{N}$; $\beta(t) = \theta_j$ if $\theta_j \leq t < \theta_{j+1},\ j \in \mathbb{Z}$, is the identification function, which was defined in [2], $\theta_j,\ j \in \mathbb{Z}$, is a strictly ordered sequence of real numbers, $|\theta_i| \to \infty$ as $|i| \to \infty$, and there exist two positive real numbers $\underline{\theta},\ \bar{\theta}$ such that $\underline{\theta} \leq \theta_{j+1} - \theta_j \leq \bar{\theta},\ j \in \mathbb{Z}$.

The study of differential equations with piecewise constant argument (EPCA) of the form

$$x'(t) = f(t, x(t), x([t])), \tag{2}$$

where $[\,.\,]$ is the greatest integer function, and extensions of the equation was initiated in [7, 24, 26]. The theory has been developed using the method of reduction to discrete equations by many authors [1, 4, 5, 6, 10, 13], [15]-[19], [22]-[23], [25], [27]-[29].

Since the greatest integer function is a particular case of the identification function, $\beta(t) = [t],\ t \in \mathbb{R}$, in otherwords $\beta(t) = i,\ i \leq t < i+1,\ i \in \mathbb{Z}$, we call systems of type (1) as differential equations with piecewise constant argument of generalized type (EPCAG).

For brief summary, the reader is referred to [8] and the book by Wiener [27]. In [2, 3], it was proposed to investigate differential equations of type (1), that is EPCAG. Moreover, a new method based on the construction of an equivalent integral equation was used.

We combine that method with the method of small parameter [12, 14, 20] to investigate the problem of the existence of a periodic solution of (1) in the so called non-critical case.

We will denote by $\|\ .\ \|$ the Euclidean norm for vectors in $\mathbb{R}^n,\ n \in \mathbb{N}$, and the uniform norm $\|C\| = \sup\{\|Cx\| \mid \|x\| = 1\}$ for $n \times n$ matrices. Let $I$ be an $n \times n$ identity matrix.

The following assumptions for equation (1) will be needed throughout the paper:

(A1) $A(t),\ f(t)$ and $g(t,x,y,\mu)$ are continuous in all of their arguments;

(A2) $g(t,x,y,\mu)$ satisfies *Lipschitz* continuity with a constant $L$ such that

$$\|g(t,\tilde{x},\tilde{y},\mu) - g(t,x,y,\mu)\| \leq L[\|\tilde{x}-x\| + \|\tilde{y}-y\|],$$

for all $t,\ \tilde{x},\ x,\ \tilde{y},\ y,\ \mu$;

The problem of the existence of periodic solutions of EPCAG in the noncritical case was previously considered in [2] by using the method of successive approximations. The aim of the present paper is to develop the notion of smoothness for solutions of EPCAG and its application to certain problems for the systems, in particular, to the method of small parameter. We plan to consider the method for critical cases in our next papers. As it is known, the method of successive approximations is complicated for critical cases. The technique of the investigation of EPCAG has also been developed by a new Lemma 3.3 in our paper.

Our paper is organized in the following way: In the next section, we consider the existence and uniqueness of a global solution of the equation defined on the real axis. In section three, continuous and differentiable dependence of the solutions on initial values and the parameter is considered. The main result of our paper: the existence of a unique $\omega$-periodic solution of the equation and its stability is investigated in the last section. Furthermore, appropriate examples are provided.

## 2 Existence and uniqueness of solutions

The following definitions are from [2]. They are similar to those in [7, 8, 17, 18, 27], adapted to EPCAG. Let us first consider solutions defined on a half line beginning at some member $\theta_i$ of sequence $\{\theta_j\},\ j \in \mathbb{Z}$.

Definition 2.1 *We say that* $x(t) = x(t, \theta_i, x_0, \mu),\ x(\theta_i) = x_0,\ i \in \mathbb{Z}$ *for* $t \geq \theta_i,\ \mu \in J,\ i \in \mathbb{Z}$, *is a solution of the initial value problem* (1) *on* $[\theta_i, \infty)$ *if it is a continuous function satisfying the conditions:*

*(a) the derivative* $x'(t)$ *exists for* $t \in [\theta_i, \infty)$ *with the possible exception of the points* $\theta_j,\ j \geq i,\ j \in \mathbb{Z}$, *where one sided derivative exists;*

*(b)* $x(t)$ *satisfies equation* (1) *for each interval* $[\theta_j, \theta_{j+1}),\ j \geq i$.

The following theorem is valid.

Theorem 2.1 *Suppose* $(A1) - (A2)$ *hold. Then, for all* $x_0 \in \mathbb{R}^n,\ \mu \in J$ *and* $i \in \mathbb{Z}$, *there exists a unique solution* $x(t)$ *of an initial value problem* (1) *with* $x(\theta_i) = x_0$ *in the sense of Definition* 2.1.

**Proof:** Let us fix $i \in \mathbb{Z},\ \mu \in J$. To use mathematical induction, let us start with $t \in [\theta_i, \theta_{i+1}]$. The solution $x(t)$ satisfies the following ordinary differential equation

$$\psi'(t) = A(t)\psi(t) + f(t) + \mu g(t, \psi(t), x_0, \mu), \tag{3}$$

$$\psi(\theta_i) = x_0, \tag{4}$$

where the functions $A(t),\ f(t)$ and $g(t, \psi(t), x_0, \mu)$ satisfy the conditions of the classical existence and uniqueness theorems of Peano and Picard-Lindelöf. Consequently, $x(t)$ exists uniquely on $[\theta_i, \theta_{i+1}]$.

Suppose that $x(t)$ is a unique solution of (1) on the interval $[\theta_i, \theta_k]$ for some $k,\ i+1 \leq k$. If $t \in [\theta_k, \theta_{k+1}]$, $x(t)$ is a solution of the following $IVP$:

$$\psi'(t) = A(t)\psi(t) + f(t) + \mu g(t, \psi(t), x(\theta_k), \mu), \tag{5}$$

$$\psi(\theta_k) = x(\theta_k). \tag{6}$$

For the same reason as that behind the existence and uniqueness of the solution of (3) and (4), we conclude that $x(t)$ is uniquely defined on this interval, too. Therefore, there exists a unique solution $x(t)$ of (1) for $t \geq \theta_i$, satisfying $x(\theta_i) = x_0$ for $x_0 \in \mathbb{R}^n$. The theorem is proved. $\square$

Let $X(t)$ be a fundamental matrix of the linear homogenous system associated with (1),

$$x'(t) = A(t)x(t), \tag{7}$$

such that $X(0) = I$, and denote $\kappa = \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$.

Lemma 2.1 *[11] Assume* (A1) *is satisfied. Then, the inequality*

$$\|X(t,s)\| \leq \exp(\kappa|t-s|),\ \ t,s \in \mathbb{R}, \tag{8}$$

*holds.*

Lemma 2.2 *Assume* (A1) *is satisfied. Then, the inequality*

$$m \leq \|X(t,s)\| \leq M,$$

*where* $m = \exp(-\kappa\bar{\theta}),\ \ M = \exp(\kappa\bar{\theta})$, *holds for* $|t-s| \leq \bar{\theta}$.

**Proof.** Using (8) and the equality $X(t,s)X(s,t) = I$, it can be found immediately that the inequality

$$\|X(t,s)\| \geq \exp(-\kappa|t-s|),\ \ t,s \in \mathbb{R}, \tag{9}$$

is satisfied. By combining (8) with (9), the lemma is proved. $\square$

The following definition is similar to those in [2, 17, 18] adapted to EPCAG.

Definition 2.2 *We say that* $x(t)$ *is a solution of* (1) *on* $\mathbb{R}$ *if it satisfies the conditions:*

*(a)* $x(t)$ *is continuous on* $\mathbb{R}$;

*(b) the derivative* $x'(t)$ *exists for all* $t \in \mathbb{R}$ *with the possible exception of the points* $\theta_j,\ \ j \in \mathbb{Z}$, *where one sided derivative exists;*

*(c)* $x(t)$ *satisfies equation* (1) *for each interval* $[\theta_j, \theta_{j+1}),\ j \in \mathbb{Z}$.

The following simple example shows that while a solution of EPCAG with small parameter exists in the sense of Definition 2.1, it may not exist in the sense of Definition 2.2, that is, a solution may exist on a half-axis and not exist on the whole real axis, unless we put some conditions.

EXAMPLE 2.1 *Consider the following differential equation:*

$$x'(t) = \alpha x(t) - \mu x^2(\beta(t)), \tag{10}$$

*where* $x \in \mathbb{R},\ t \in \mathbb{R},\ \ \alpha > 0$ *is a constant, and* $\beta(t) = \theta_j$ *if* $\theta_j \le t < \theta_{j+1},\ j \in \mathbb{Z},\ \theta_{2i-1} = 4i - 1,\ \theta_{2i} = 4i,\ i \in \mathbb{Z}$. *The distance* $\theta_{j+1} - \theta_j,\ j \in \mathbb{Z}$, *is either equal to* $\underline{\theta} = 1$ *or to* $\bar{\theta} = 3$. *Let us fix* $x_0 \in \mathbb{R}$. *We shall look for conditions on* $\alpha$ *and* $\mu$ *such that a solution* $x(t) = x(t, \theta_0, x_0, \mu),\ x(\theta_0) = x_0,\ x_0 > 0$, *of* (10) *exists in the sense of Definitions* 2.1 *and* 2.2.

*If* $\mu = 0$, *it is easy to see that the solution* $x(t)$ *of* (10) *exists uniquely, and it is positive and not bounded on* $\mathbb{R}$.

*Suppose* $\mu > 0$. *Let us consider a transformation* $x(t) = \frac{y(t)}{\mu}$. *Using this transformation, we obtain the following equation from* (10)

$$y'(t) = \alpha y(t) - y^2(\beta(t)). \tag{11}$$

*Let* $y(t) = y(t, \theta_0, y_0)$ *be a solution of* (11) *with* $y(\theta_0) = y_0,\ y_0 > 0$. *Denote* $y_k = y(\theta_k),\ k \in \mathbb{Z}$. *We first consider the existence and uniqueness of the solution* $y(t)$. *Let us start with* $t \in [\theta_0, \infty)$, *that is, if time is increasing.*

*If* $t \in [\theta_0, \theta_1]$, *then* $y(t)$ *is a solution of the equation*

$$y'(t) = \alpha y(t) - y_0^2,$$

*which is a linear nonhomogeneous differential equation with a constant coefficient, that is why the solution* $y(t)$ *is uniquely defined on* $[\theta_0, \theta_1]$. *The rest can be deduced* *from the arguments of mathematical induction. That is, the solution $y(t)$ and the corresponding solution $x(t)$ exist uniquely on $[\theta_0, \infty)$ in the sense of Definition* 2.1.

*Next, let us consider the solution for decreasing time. We will show that if*

$$\begin{aligned} y_0 &\leq \frac{\alpha \exp(2\alpha)}{4[\exp(\alpha) - 1]} \qquad (12) \\ \frac{\alpha \exp(\alpha)}{\exp(\alpha) - 1} &\leq \frac{\alpha \exp(6\alpha)}{4[\exp(3\alpha) - 1]} \qquad (13) \\ \frac{\alpha \exp(3\alpha)}{\exp(3\alpha) - 1} &\leq \frac{\alpha \exp(2\alpha)}{4[\exp(\alpha) - 1]} \qquad (14) \end{aligned}$$

*are satisfied, then the solution $y(t) = y(t, \theta_0, y_0)$ exists on $(-\infty, \theta_0]$.*

*If $t \in [\theta_{-1}, \theta_0]$, then $y(t)$ coincides with the solution of the following ordinary differential equation*

$$y'(t) = \alpha y(t) - y_{-1}^2. \qquad (15)$$

*Using the equivalent integral equation of* (15), *it can be written that*

$$y(t) = \exp(\alpha(t - \theta_{-1}))y_{-1} + \frac{1}{\alpha}[1 - \exp(\alpha(t - \theta_{-1}))]y_{-1}^2. \qquad (16)$$

*Denote $z = y_{-1}$. It is easy to see that the solution $y(t)$ exists on $[\theta_{-1}, \theta_0]$, if the quadratic equation for $z$, obtained from* (16) *with $t = \theta_0$,*

$$z^2 - \frac{\alpha \exp(\alpha)}{\exp(\alpha) - 1} z + \frac{\alpha}{\exp(\alpha) - 1} y_0 = 0 \qquad (17)$$

*has a real root. The last equation has a real root, if inequality* (12) *is valid. Hence, if inequality* (12) *is valid, then the solution $y(t)$ exists on $[\theta_{-1}, \theta_0]$, but is not necessarily unique.*

*Suppose inequality* (12) *is valid. It is easy to check that the roots $z_{1,2}$ of equation* (17) *satisfy the inequality*

$$0 \leq z_{1,2} \leq \frac{\alpha \exp(\alpha)}{\exp(\alpha) - 1}. \qquad (18)$$

*Denote* $z = y_{-2}$. *If* $t \in [\theta_{-2}, \theta_{-1}]$, *one can similarly obtain that the solution* $y(t)$ *exists on* $[\theta_{-2}, \theta_{-1}]$, *if the following quadratic equation*

$$z^2 - \frac{\alpha \exp(3\alpha)}{\exp(3\alpha) - 1} z + \frac{\alpha}{\exp(3\alpha) - 1} y_{-1} = 0 \tag{19}$$

*has a real root. The last equation has a real root, if*

$$y_{-1} \leq \frac{\alpha \exp(6\alpha)}{4[\exp(3\alpha) - 1]} \tag{20}$$

*holds. Using inequalities* (18) *and* (20), *it is clear that if inequality* (13) *is valid, then the solution* $y(t)$ *exists on* $[\theta_{-2}, \theta_{-1}]$.

*Suppose inequality* (13) *is valid. It is easy to see that the roots* $z_{3,4}$ *of equation* (19) *satisfy the inequality*

$$0 \leq z_{3,4} \leq \frac{\alpha \exp(3\alpha)}{\exp(3\alpha) - 1}. \tag{21}$$

*If* $t \in [\theta_{-3}, \theta_{-2}]$, *we then have a quadratic equation similar to* (17), *and*

$$y_{-2} \leq \frac{\alpha \exp(2\alpha)}{4[\exp(\alpha) - 1]} \tag{22}$$

*holds. Therefore, the solution* $y(t)$ *exists on* $[\theta_{-3}, \theta_{-2}]$. *Finally, using inequalities* (21), (22) *and* (14) *one can see that the solution* $y(t)$ *exists on* $[\theta_{-3}, \theta_{-2}]$.

*By using the arguments of mathematical induction, we can conclude that if inequalities* $(12)-(14)$ *are satisfied, then the solution* $y(t, \theta_0, y_0)$ *exists on* $(-\infty, \theta_0]$, *but is not necessarily unique.*

*Consequently, if inequalities* (13), (14) *and the inequality*

$$0 < \mu \leq \frac{\alpha \exp(2\alpha)}{4x_0[\exp(\alpha) - 1]}, \tag{23}$$

*obtained from* (12), *are satisfied for* $x_0 > 0$, *then the solution* $x(t) = x(t, \theta_0, x_0, \mu)$ *exists in the sense of Definition* 2.2. *Moreover, if one of inequalities* (13), (14) *or* (23) *is violated, then the solution* $x(t)$ *exists in the sense of Definition* 2.1, *but it doesn't exist in the sense of Definition* 2.2.

From now on, we make the following assumptions:

(A3) $|\mu| < 1/(2ML\bar{\theta})$,

(A4) $|\mu|ML\bar{\theta}[1 + M(1 + L|\mu|\bar{\theta})\exp(ML|\mu|\bar{\theta})] < m$.

The following theorem provides the existence of a unique solution to the left when the initial moment $\xi$ is an arbitrary real number.

THEOREM 2.2 *Suppose* (*A1*) $-$ (*A4*) *hold. Then, for all* $x_0 \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $\theta_i < \xi \leq \theta_{i+1}$, $i \in \mathbb{Z}$, *there exists a unique solution* $\bar{x}(t) = x(t, \theta_i, \bar{x}_0, \mu)$ *of* (1) *in the sense of Definition* 2.1 *with* $\bar{x}(\xi) = x_0$.

**Proof:** *Existence.* Consider a solution $\psi(t) = x(t, \xi, x_0, \mu)$ with $\psi(\xi) = x_0$ of the equation

$$x'(t) = A(t)x(t) + f(t) + \mu g(t, x(t), \eta, \mu)$$

on $[\theta_i, \xi]$.

We need to prove that there is a vector $\eta \in \mathbb{R}^n$ such that the equation

$$\psi(t) = X(t, \xi)x_0 + \int_{\xi}^{t} X(t, s)[f(s) + \mu g(s, \psi(s), \eta, \mu)]\, ds, \tag{24}$$

has a solution $\psi(t)$, defined on $[\theta_i, \xi]$, and satisfying $\psi(\theta_i) = \eta$.

Construct a sequence $\{\psi_k(t)\} \subset \mathbb{R}^n$, $k \in \mathbb{N}$ with $\psi_0(t) = X(t, \xi)x_0$ such that

$$\psi_{k+1}(t) = X(t, \xi)x_0 + \int_{\xi}^{t} X(t, s)[f(s) + \mu g(s, \psi_k(s), \psi_k(\theta_i), \mu)]\, ds, \ k \in \mathbb{N}.$$

By simple calculation, it can be found that

$$\max_{[\theta_i, \xi]} \|\psi_{k+1}(t) - \psi_k(t)\| \leq (2ML\bar{\theta}|\mu|)^k \zeta,$$

where $\zeta = M\bar{\theta} \max_{[\theta_i, \xi]} \|f(s) + \mu g(s, \psi_0(s), \psi_0(\theta_i), \mu)\|$. That is, the sequence $\psi_k(t)$ is convergent and its limit $\psi(t)$ satisfies (24) on $[\theta_i, \xi]$ with $\eta = \psi(\theta_i)$ whenever $|\mu| < 1/(2ML\bar{\theta})$. The existence is proved.

*Uniqueness.* It is sufficient to check that for each $t \in (\theta_i, \theta_{i+1}]$, and $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, condition $x(t, \theta_i, x_1, \mu) \neq x(t, \theta_i, x_2, \mu)$ is valid.

Let us denote by $x_1(t) = x(t, \theta_i, x_1, \mu)$, $x_2(t) = x(t, \theta_i, x_2, \mu)$, $x_1 \neq x_2$, solutions of (1). Assume to the contrary that there exists $\tilde{t} \in (\theta_i, \theta_{i+1}]$ such that $x_1(\tilde{t}) = x_2(\tilde{t})$, which implies the equation

$$X(\tilde{t}, \theta_i)(x_1 - x_2) = -\mu \int_{\theta_i}^{\tilde{t}} X(\tilde{t}, s)[g(s, x_1(s), x_1(\theta_i), \mu) - g(s, x_2(s), x_2(\theta_i), \mu)]\, ds. \tag{25}$$

We have the inequalities

$$\|X(\tilde{t}, \theta_i)(x_1 - x_2)\| \geq m\|x_1 - x_2\| \tag{26}$$

and

$$\|x_1(t) - x_2(t)\| \leq M\|x_1 - x_2\| + \int_{\theta_i}^{t} ML\ |\mu|[\|x_1(s) - x_2(s)\| + \|x_1 - x_2\|]\, ds,$$

for $t \in (\theta_i, \theta_{i+1}]$. Hence, by applying Gronwall Bellman Lemma to the last inequality, we can write

$$\|x_1(t) - x_2(t)\| \leq M(1 + L|\mu|\bar{\theta}) \exp(ML|\mu|\bar{\theta})\|x_1 - x_2\|.$$

Consequently,

$$\left\| -\mu \int_{\theta_i}^{\tilde{t}} X(\tilde{t}, s)[g(s, x_1(s), x_1(\theta_i), \mu) - g(s, x_2(s), x_2(\theta_i), \mu)]\, ds \right\| \leq$$
$$|\mu|ML\bar{\theta}[1 + M(1 + L|\mu|\bar{\theta}) \exp(ML|\mu|\bar{\theta})]\|x_1 - x_2\|. \tag{27}$$

Finally, one can see that $(A4)$, (26) and (27) contradict (25). The theorem is proved. □

*Remark 2.1 The last theorem provides us conditions,* $(A3)$ *and* $(A4)$, *of smallness for the parameter* $\mu$ *such that the initial value problem has a unique solution defined on* $[t_0, \infty)$.

The following theorem is valid.

Theorem 2.3 *Suppose* $(A1)-(A4)$ *hold. Then, for all* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, *there exists a unique solution* $x(t)$ *of* (1) *in the sense of Definition* 2.2 *with* $x(t_0) = x_0$.

**Proof:** Take a moment $t_0 \in \mathbb{R}$. Then, there is $i \in \mathbb{Z}$ such that $\theta_i < t_0 \leq \theta_{i+1}$. By Theorem 2.2, there is a unique solution $x(t) = x(t, \theta_i, {x_0}^i, \mu),\ x(\theta_i) = {x_0}^i$ of (1) with $x(t_0) = x_0$. Similarly, by Theorem 2.2, there is a unique solution $\tilde{x}(t) = x(t, \theta_{i-1}, {x_0}^{i-1}, \mu),\ \tilde{x}(\theta_{i-1}) = {x_0}^{i-1}$ with $\tilde{x}(\theta_i) = {x_0}^i$. Hence, $\tilde{x}(t_0) = x_0$. We can complete the proof by using mathematical induction. $\square$

The last theorem is of major importance, since it supplies a one-to-one correspondence between points $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and solutions of (1), and there is no solution of (1) out of the correspondence. Although (1) is a delay differential equation, it has the properties of ordinary differential equations. We will make use of this correspondence in the rest of the paper.

# 3 Dependence of solutions on the initial value and the parameter

Let us introduce the following two lemmas. We prove only the second of them, the proof of the first one is very similar.

Lemma 3.1 *Suppose* $(A1)$ *is satisfied. A function* $x(t) = x(t, t_0, x_0, \mu)$, *where* $t_0$ *is a real fixed number, is a solution of* (1) *on* $\mathbb{R}$ *if and only if it is a solution on* $\mathbb{R}$ *of the following integral equation*

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)[f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds. \tag{28}$$

Lemma 3.2 *Suppose* $(A1)$ *is satisfied. A function* $x(t) = x(t, t_0, x_0, \mu)$, *where* $t_0$ *is a real fixed number, is a solution of* (1) *on* $\mathbb{R}$ *if and only if it is a solution on* $\mathbb{R}$ *of the following integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds. \tag{29}$$

**Proof:** *Necessity.* Assume that $x(t)$ is a solution of (1) on $\mathbb{R}$. Denote

$$\phi(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds.$$

By straightforward evaluation, we can see that the integral exists.

Assume that $t \neq \theta_i,\ i \in \mathbb{Z}$. Then

$$\phi'(t) \quad = \quad A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu)$$

and

$$x'(t) \quad = \quad A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu).$$

Hence,

$$[\phi(t) - x(t)]' \quad = \quad 0.$$

Calculating the limit values at $\theta_i,\ i \in \mathbb{Z}$, we find that

$$\phi'(\theta_i \pm 0) = A(\theta_i \pm 0)x(\theta_i \pm 0) + f(\theta_i \pm 0) + \mu g(\theta_i \pm 0, x(\theta_i \pm 0), x(\beta(\theta_i \pm 0)), \mu),$$

$$x'(\theta_i \pm 0) = A(\theta_i \pm 0)x(\theta_i \pm 0) + f(\theta_i \pm 0) + \mu g(\theta_i \pm 0, x(\theta_i \pm 0), x(\beta(\theta_i \pm 0)), \mu).$$

Consequently,

$$[\phi(t) - x(t)]'|_{t=\theta_i+0} = [\phi(t) - x(t)]'|_{t=\theta_i-0}.$$

Thus, $\phi(t) - x(t)$ is a continuously differentiable function on $\mathbb{R}$ satisfying the equation

$$[\phi(t) - x(t)]' = 0 \tag{30}$$

with the initial condition $\phi(t_0) - x(t_0) = 0$. This proves that $\phi(t) - x(t) = 0$ on $\mathbb{R}$.

*Sufficiency.* Suppose that (29) is valid. Fix $i \in \mathbb{Z}$ and consider the interval $[\theta_i, \theta_{i+1})$. If $t \in (\theta_i, \theta_{i+1})$, then by differentiating (29) one can see that $x(t)$ satisfies

(1). Moreover, by considering $t \to \theta_i+$, and taking into account that $x(\beta(t))$ is a right continuous function, we find that $x(t)$ satisfies (1) on $[\theta_i, \theta_{i+1})$. The lemma is proved. $\square$

Let us denote by $\| \,.\, \|_t$ a sup-norm, $\|v(\xi)\|_t = \sup_{[\theta_0,t]} \|v(\xi)\|$. The next two theorems set continuous dependence of solutions for (1) on a parameter and an initial value. To prove the theorems, we consider the following assertion.

Lemma 3.3 *Let $v(t)$ be a continuous function for $t \ge \theta_0$, satisfying*

$$\|v(t)\| \le \alpha + \int_{\theta_0}^{t} [\zeta_1(s)\,\|v(s)\| + \zeta_2(s)\,\|v(\beta(s))\|]\,ds, \tag{31}$$

*where $\alpha \ge 0$ and $\zeta_1(t),\ \zeta_2(t)$ are nonnegative piecewise continuous functions. Then,*

$$\|v(\xi)\|_t \le \alpha \exp\left(\int_{\theta_0}^{t} [\zeta_1(s) + \zeta_2(s)]\,ds\right),\ t \ge \theta_0. \tag{32}$$

**Proof:** Let us first show that

$$\|v(\xi)\|_t \le \alpha + \int_{\theta_0}^{t} [\zeta_1(s) + \zeta_2(s)]\ \|v(\xi)\|_s\,ds,\ t \ge \theta_0. \tag{33}$$

Since $\theta_0 \le \beta(s)) \le s$ for $s \ge \theta_0$, we have that

$$\|v(\beta(\xi))\|_t = \sup_{[\theta_0,t]} \|v(\beta(\xi))\| = \sup_{[\theta_0,\beta(t)]} \|v(\xi)\| \le \sup_{[\theta_0,t]} \|v(\xi)\| = \|v(\xi)\|_t.$$

Hence,

$$\|v(t)\| \le \alpha + \int_{\theta_0}^{t} [\zeta_1(s) + \zeta_2(s)]\ \|v(\xi)\|_s\,ds$$

is satisfied.

If $\|v(t)\| = \|v(\xi)\|_t$ for a given $t \ge \theta_0$, then inequality (33) is valid. Suppose that $\|v(t)\| < \|v(\xi)\|_t$. Then, by definition of sup-norm, there is a moment $\tilde{t} \in [\theta_0, t]$ such that $\|v(\tilde{t})\| = \|v(\xi)\|_t$. Hence,

$$\begin{aligned} \|v(\xi)\|_t &= \|v(\tilde{t})\| \\ &\le \alpha + \int_{\theta_0}^{\tilde{t}} [\zeta_1(s) + \zeta_2(s)]\ \|v(\xi)\|_s\,ds \\ &\le \alpha + \int_{\theta_0}^{t} [\zeta_1(s) + \zeta_2(s)]\ \|v(\xi)\|_s\,ds, \end{aligned}$$

as $\tilde{t} \leq t$. So, (33) is valid. Now, setting $\psi(t) = \|v(\xi)\|_t$ and applying Gronwall Bellman Lemma to

$$\psi(t) \leq \alpha + \int_{\theta_0}^{t} [\zeta_1(s) + \zeta_2(s)] \ \psi(s)\, ds, \ t \geq \theta_0,$$

we complete the proof. $\square$

Let us fix a number $T > 0$.

THEOREM 3.1 *Suppose* $(A1) - (A4)$ *are valid. If* $x(t) = x(t, \theta_0, x_0, \mu_0)$ *and* $\tilde{x}(t) = x(t, \theta_0, x_0, \mu_0 + \delta)$ *are solutions of* (1) *on* $[\theta_0, \theta_0 + T]$, *then*

$$\|\tilde{x}(\xi) - x(\xi)\|_t \leq \delta GT \exp(2(\kappa + |\mu_0|L)T), \ \theta_0 \leq t \leq \theta_0 + T, \tag{34}$$

*where*

$$G = \sup_{(t,\ \mu) \in [\theta_0, \theta_0 + T] \times J} \|g(t, x, y, \mu)\|,$$

*is satisfied for* $t \in [\theta_0, \theta_0 + T]$.

**Proof:** By straightforward evaluation, if $t \in [\theta_0, \theta_0 + T]$, we obtain

$$\begin{aligned} \|\tilde{x}(t) - x(t)\| \ &\leq \ \int_{\theta_0}^{t} [\|A(s)\| \, \|\tilde{x}(s) - x(s)\| + \delta \, \|g(s, \tilde{x}(s), \tilde{x}(\beta(s)), \mu + \delta)\| \\ &\quad + |\mu_0| \, \|g(s, \tilde{x}(s), \tilde{x}(\beta(s)), \mu + \delta) - g(s, x(s), x(\beta(s)), \mu)\|] \, ds. \end{aligned}$$

Then,

$$\|\tilde{x}(t) - x(t)\| \leq GT\delta + (\kappa + |\mu_0|L) \int_{\theta_0}^{t} [\|\tilde{x}(s) - x(s)\| + \|\tilde{x}(\beta(s)) - x(\beta(s))\|] \, ds.$$

Applying Lemma 3.3 to the last inequality, we proved that (34) is valid. $\square$

THEOREM 3.2 *Suppose* $(A1) - (A4)$ *are valid. If* $x(t) = x(t, \theta_0, x_0, \mu_0)$ *and* $\tilde{x}(t) = x(t, \theta_0, x_0 + \Delta x, \mu_0)$ *are the solutions of* (1), *where* $\Delta x$ *is an* $n$*-dimensional vector, then*

$$\|\tilde{x}(\xi) - x(\xi)\|_t \leq \|\Delta x\| \exp((\kappa + 2|\mu|L)T) \tag{35}$$

*is satisfied for* $t \in [\theta_0, \theta_0 + T]$.

**Proof:** If $t \in [\theta_0, \theta_0 + T]$, then

$$\begin{aligned}\|\tilde{x}(t) - x(t)\| &\leq \|\Delta x\| + \int_{\theta_0}^{t} [\|A(s)\| \|\tilde{x}(s) - x(s)\| + |\mu| \|g(s, \tilde{x}(s), \tilde{x}(\beta(s)), \mu) \\ &\quad - g(s, x(s), x(\beta(s)), \mu)\|] \, ds,\end{aligned}$$

and

$$\|\tilde{x}(t) - x(t)\| \leq \|\Delta x\| + (\kappa + |\mu| L) \int_{\theta_0}^{t} [\|\tilde{x}(s) - x(s)\| + \|\tilde{x}(\beta(s)) - x(\beta(s))\|] \, ds.$$

Applying Lemma 3.3 to the last inequality, we proved that (35) is valid. $\square$

The differential dependence of a solution of (1) on an initial value is established by our next theorem. We need the following assumption:

(A5) $g(t, x, y, \mu)$ has continuous first partial derivatives in all its arguments $t \in \mathbb{R},\ x, y \in \mathbb{R}^n,\ \mu \in J.$

Let us introduce the following equations:

$$U'(t) = A_1(t) U(t) + A_2(t) U(\beta(t)), \tag{36}$$

$$U(\theta_0) = I, \tag{37}$$

where

$$A_1(t) = A(t) + \mu \frac{\partial g}{\partial x}(t, x(t), x(\beta(t)), \mu),\ A_2(t) = \mu \frac{\partial g}{\partial y}(t, x(t), x(\beta(t)), \mu)$$

are $n \times n$ matrices.

THEOREM 3.3 *Suppose* $(A1) - (A5)$ *are valid. Let* $e_j$ *be the* $j$*th standard basis* $n$*-dimensional vector, and* $U(t)$ *the solution on* $\mathbb{R}$ *of* (36) *and* (37) *in the sense of Definition* 2.2. *If* $x(t) = x(t, \theta_0, x_0, \mu_0)$ *and* $\tilde{x}_j(t) = x(t, \theta_0, x_0 + \Delta x_j, \mu_0)$ *are the solutions of* (1), *where* $\Delta x_j = e_j \Delta x$ *is an* $n$*-dimensional vector, then*

$$\tilde{x}_j(t) - x(t) - U(t) \Delta x_j = o(\Delta x_j). \tag{38}$$

*is satisfied for* $t \in [\theta_0, \theta_0 + T]$.

**Proof:** By the equivalence Lemma 3.2, $\tilde{x}_j(t)$, $x(t)$ and $U(t)$ satisfy the following integral equations:

$$\begin{aligned}
\tilde{x}_j(t) &= x_0 + \Delta x_j + \int_{\theta_0}^{t} [A(s)\tilde{x}_j(s) + f(s) + \mu g(s, \tilde{x}_j(s), \tilde{x}_j(\beta(s)), \mu)]\, ds, \\
x(t) &= x_0 + \int_{\theta_0}^{t} [A(s)x(s) + f(s) + \mu g(s, x(s), x(\beta(s)), \mu)]\, ds, \\
U(t) &= I + \int_{\theta_0}^{t} [A_1(s)U(s) + A_2(s)U(\beta(s))]\, ds,
\end{aligned}$$

respectively. An easy computation shows that, if $t \in [\theta_0, \theta_0 + T]$,

$$\begin{aligned}
\tilde{x}_j(t) - x(t) - U(t)\Delta x_j = \int_{\theta_0}^{t} [A(s)(\tilde{x}_j(s) - x(s)) + \mu[g(s, \tilde{x}_j(s), \tilde{x}_j(\beta(s)), \mu) \\
-g(s, x(s), x(\beta(s)), \mu)] - A_1(s)U(s)\Delta x_j - A_2(s)U(\beta(s))\Delta x_j]\, ds.
\end{aligned}$$

By expanding $g(t, \tilde{x}_j(t), \tilde{x}_j(\beta(t)), \mu)$ about $(t, x(t), x(\beta(t)), \mu)$, we write

$$\begin{aligned}
A(s)(\tilde{x}_j(s) - x(s)) + \mu g(t, \tilde{x}_j(t), \tilde{x}_j(\beta(t)), \mu) = \mu g(t, x(t), x(\beta(t)), \mu) \\
+A_1(t)(\tilde{x}_j(t) - x(t)) + A_2(t)(\tilde{x}_j(\beta(t)) - x(\beta(t))) + \xi(t),
\end{aligned}$$

where $\xi(t) = o(\Delta x_j, t)$. Hence,

$$\begin{aligned}
\|\tilde{x}_j(t) - x(t) - U(t)\Delta x_j\| \le \zeta + \int_{\theta_0}^{t} [\|A_1(s)\|\ \|\tilde{x}_j(s) - x(s) - U(s)\Delta x_j\| \\
+\|A_2(s)\|\ \|\tilde{x}_j(\beta(s)) - x(\beta(s)) - U(\beta(s))\Delta x_j\|]\, ds,
\end{aligned}$$

where $\zeta = \int_{\theta_0}^{\theta_0+T} \|\xi(s)\|\, ds$. Consequently, by applying Lemma 3.3 to the last inequality and using $\zeta = o(\Delta x_j)$, we proved that (38) is valid. $\square$

As a result of the last theorem, we have shown that the initial value problem (36) and (37) is a variation of (1).

# 4 Existence and stability of periodic solutions

In this section, we prove the main theorem of this paper. We need the following assumptions:

(A6) $A(t),\ f(t)$ and $g(t,x,y,\mu)$ are periodic in $t$ with a fixed positive real period $\omega$;

(A7) the sequence $\{\theta_j\}$ satisfies an $(\omega,p)$-property, that is $\theta_{j+p}=\theta_j+\omega,\ j\in\mathbb{Z}$, for some positive integer $p$.

Let us consider the following version of the Poincaré criterion.

LEMMA 4.1 *Suppose* $(A1)-(A4)$ *and* $(A6)-(A7)$ *hold. Then, solution* $x(t)=x(t,t_0,x_0,\mu)$ *of* $(1)$, *with* $x(t_0)=x_0$, *is* $\omega$*-periodic if and only if*

$$x(\omega)=x(0). \tag{39}$$

**Proof.** If $x(t)$ is $\omega$-periodic, then equation (39) is obviously satisfied. Suppose equation (39) holds. Let $y(t)=x(t+\omega)$ on $\mathbb{R}$. Then, equation (39) can be written as $y(0)=x(0)$. One can show that $\beta(t+\omega)=\beta(t)+\omega$. Hence,

$$\begin{aligned} y'(t) &= x'(t+\omega)\\ &= A(t+\omega)x(t+\omega)+f(t+\omega)+\mu g(t+\omega,x(t+\omega),x(\beta(t+\omega)),\mu)\\ &= A(t)y(t)+f(t)+\mu g(t,y(t),y(\beta(t)),\mu). \end{aligned}$$

That is, $y(t)$ is a solution of (1). By uniqueness of the solution, $x(t)=y(t)$ on $\mathbb{R}$. The lemma is proved. $\square$

The following theorem is a generalization of a classical theorem orginally due to Poincaré [20] for EPCAG. The proof of this theorem can be found in [9, p. 67] and [12].

THEOREM 4.1 *Assume that* $(A1)-(A7)$ *hold, and*

$$x'(t)=A(t)x(t) \tag{40}$$

*has no periodic solution with period* $\omega$. *Then, for sufficiently small* $\mu$, *equation* (1) *has a unique* $\omega$*-periodic solution, which tends to the unique periodic solution* *with period $\omega$ of*

$$x'(t) = A(t)x(t) + f(t), \tag{41}$$

*as* $\mu \to 0$.

**Proof:** Let $x(t, \zeta, \mu)$ be a solution of equation (1), satisfying the initial condition $x(0, \zeta, \mu) = \zeta$, and let $x_0(t) = x(t, \zeta_0, 0)$ be a unique periodic solution of period $\omega$ of equation (41). To show, using Lemma 4.1, that for a sufficiently small $\mu$ the $\omega$-periodic solution $x(t, \zeta, \mu)$ exists, it is necessary and sufficient that the equation

$$x(\omega, \zeta, \mu) - \zeta = 0 \tag{42}$$

be solvable with respect to $\zeta$.

Let $P(\zeta, \mu) = x(\omega, \zeta, \mu) - \zeta$. In order to apply the implicit function theorem, we show that the determinant of $P_\zeta{}'(\zeta_0, 0)$ exists and is different from zero.

Let $Z(t, \zeta, \mu) = (\partial x_j / \partial \zeta_k),\ j,\ k = 1, \ldots, n$. Differentiating equation (1) with respect to $\zeta$, we can see that $Z(t, \zeta_0, 0)$ is the fundamental matrix of equation (40). On the other hand, $P_\zeta{}'(\zeta_0, 0) = \det(Z(\omega, \zeta_0, 0) - I)$ and, since the eigenvalues of the matrix $Z(\omega, \zeta_0, 0)$ are different from unity, it follows that $P_\zeta{}'(\zeta_0, 0) \neq 0$. Therefore, in a sufficiently small neighborhood of the point $(0, \zeta_0)$, equation (42) is solvable with respect to $\zeta$. The existence and uniqueness of an $\omega$-periodic solution are proved. The fact that the solution $x(t, \zeta, \mu)$ tends to $x_0(t)$ when $\mu \to 0$ follows from Theorem 3.1. The theorem is proved. $\square$

Let us demonstrate the last theorem by applying it to the following example.

Example 4.1 *Consider the following system of EPCAG:*

$$x'(t) = \begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix} x(t) + \begin{pmatrix} sin(\pi t) \\ cos(\pi t) \end{pmatrix} + \mu\ g(t, x(t), x(\beta(t)), \mu), \tag{43}$$

*where* $x \in \mathbb{R}^2$, $\alpha \neq 0$, $\gamma, \mu \geq 0$, $\beta(t) = \theta_i$ *if* $\theta_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$ *with* $\theta_j = j + \frac{(-1)^j}{3}$, $j \in \mathbb{Z}$; $g(t, x, y, \mu)$ *is a* $2-$*periodic in* $t$, *continuous function, having continuous first partials in all of its arguments, and satisfying Lipschitz continuity with a constant* $L$, *that is,*

$$\|g(t, x_1, y_1, \mu) - g(t, x_2, y_2, \mu)\| \leq L\left[\|x_1 - x_2\| + \|y_1 - y_2\|\right],$$

*where* $x_1, y_1, x_2, y_2 \in \mathbb{R}^2$. *One can see that the sequence* $\{\theta_i\}$ *fulfills* $\theta_{i+2} = \theta_i + 2$ *for all* $i \in \mathbb{Z}$. *By fixing a sufficiently small* $\mu$ *satisfying the inequalities*

$$|\mu| < 1/(2ML\bar{\theta}),$$
$$|\mu| ML\bar{\theta}[1 + M(1 + L|\mu|\bar{\theta})\exp(ML|\mu|\bar{\theta})] < m,$$

*where* $\bar{\theta} = 5/3$, $\kappa = \sqrt{\alpha^2 + \gamma^2}$, $M = e^{5\kappa/3}$ *and* $m = e^{-5\kappa/3}$, *we conclude that assumptions* $(A1) - (A7)$ *are fulfilled. Therefore, through every point* $(t_0, \zeta)$ *of* $\mathbb{R}^3$, *there passes exactly one solution* $x(t, \mu) = x(t, t_0, \zeta, \mu)$, $x(t_0, \mu) = \zeta$ *of* (43) *in the sense of Definition* 2.2.

*The monodromy matrix of* (43) *is*

$$X(2) = \begin{bmatrix} e^{2\alpha}\cos(2\gamma) & e^{2\alpha}\sin(2\gamma) \\ -e^{2\alpha}\sin(2\gamma) & e^{2\alpha}\cos(2\gamma) \end{bmatrix},$$

*and it has no unit multiplier for* $\alpha \neq 0$. *Hence, there is a unique 2-periodic solution* $x_0(t)$ *of the system*

$$x'(t) = \begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix} x(t) + \begin{pmatrix} sin(\pi t) \\ cos(\pi t) \end{pmatrix},$$

*with the initial value*

$$x_0(\theta_0) = (I - X(2))^{-1} \int_{\theta_0}^{\theta_2} \left[ X(\theta_2 - s) \begin{pmatrix} sin(\pi s) \\ cos(\pi s) \end{pmatrix} \right] ds.$$

*Therefore, by Theorem* 4.1, *there is a unique 2-periodic solution* $x(t, \mu)$ *of* (43), *satisfying* $x(t, \mu) \to x_0(t)$ *as* $\mu \to 0$.

Definition 4.1 *The solution* $x(t,x_0)=x(t,x_0,\mu),\ x(\theta_0,x_0)=x_0$ *of* (1) *is said to be stable if for every* $\epsilon>0$, *there exists a number* $\delta\in\mathbb{R},\ \delta=\delta(\epsilon,\theta_0)>0$ *such that* $\|\zeta-x_0\|<\delta$ *implies* $\|x(t,\zeta)-x(t,x_0)\|<\epsilon$ *for all* $t\geq\theta_0$.

Definition 4.2 *The solution* $x(t,x_0)=x(t,x_0,\mu),\ x(\theta_0,x_0)=x_0$ *of* (1) *is said to be asymptotically stable if it is stable and there exists a number* $\eta\in\mathbb{R},\ \eta=\eta(\theta_0)>0$ *such that* $\|\zeta-x_0\|<\eta$ *implies* $\lim_{t\to\infty}\|x(t,\zeta)-x(t,x_0)\|=0$.

Theorem 4.2 *Suppose that* (A1)−(A7) *hold. Let* $x(t)=x(t,x_0,\mu)$ *be a solution of* (1) *with* $x(\theta_0)=x_0$. *If all the multipliers of the equation*

$$x'(t)=A(t)x(t) \tag{44}$$

*are less than unity in absolute value, then for sufficiently small* $\mu$, *the solution* $x(t)$ *is asymptotically stable.*

**Proof.** Let $u(t)$ be a solution of (1) with the initial condition $u(\theta_0)=x_0+\eta$. Let us define $z(t)=u(t)-x(t)$. Since all the multipliers are less than unity in absolute value,

$$\|X(t,s)\|\leq K\exp(-\alpha(t-s)),\ \ s\leq t,$$

where $K$ and $\alpha$ are positive constants. By using the equivalence Lemma 3.1, one can find that

$$\|z(t)\|\leq\|X(t,\theta_0)\|\|\eta\|+\int_{\theta_0}^{t}\|X(t,s)\|\ |\mu|[\|g(s,x(s)+z(s),x(\beta(s))+z(\beta(s)),\mu) \\ -g(s,x(s),x(\beta(s)),\mu)\|]\,ds$$

and

$$\|z(t)\|\leq K\exp(-\alpha(t-\theta_0))\|\eta\|+\int_{\theta_0}^{t}\exp(-\alpha(t-s))\ |\mu|KL[\|z(s)\|+\|z(\beta(s))\|]\,ds.$$

Then,

$$\exp(\alpha t)\|z(t)\|\leq K\exp(\alpha\theta_0)\|\eta\|+\int_{\theta_0}^{t}\exp(\alpha s)\ |\mu|KL[\|z(s)\|+\|z(\beta(s))\|]\,ds.$$

Applying Lemma 3.3 to the last inequality, we have

$$\|z(t)\| \leq K\exp([-\alpha + 2|\mu|KL](t-\theta_0))\|\eta\|$$

Therefore, for $|\mu| < \alpha/(2KL)$, solution $q(t)$ is asymptotically stable. The theorem is proved. $\square$

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