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An impulsive ratio-dependent predator–prey system with diffusion

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Abstract

We investigate the predator–prey system with diffusion, when biological and environmental parameters are assumed to change in periodical manner over time. The system is affected by impulses which can be considered as a control. Conditions for the permanence of the predator–prey system and for the existence of a unique globally stable periodic solutions are obtained.

Keywords: Predator-prey system with diffusion; Functional response; Permanence; Periodic solutions; Stability

1. Introduction

On large scale distribution, individuals of populations are referred to as local populations that are distributed within a given habitat patch with a pattern of clumped distribution. Clumping or aggregation of the populations may result from the social predisposition of individuals to form groups, clumped distribution of resources, and a tendency of progeny to remain in the vicinity of their parent [16]. Furthermore, distribution of individuals of populations can also follow the same random and regular patterns.

The pattern may be a result of an interaction between the spatial arrangement of habitat patches and other ecological or behavioral processes. Stabilization of predator–prey dynamics can be achieved through spatial heterogeneity, whose stabilizing effect of partial isolation of habitat patches is very important [20].

In this study we neglect the impact of migration for effecting both prey and predator populations. To meet such condition, we assumed that our prey and predator populations are found in an isolated habitat for which the impact of migration, including both emigration and immigration, is presumably negligible, such as a remote patchy forest or an isolated island or a lake ecosystem which is practically water islands with distinct boundaries.

Holling [9] introduced the concept of the functional response, which described the asymptotic relationship between prey removal rate per predator and the density of prey. Functional responses describe that number of prey consumed

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increases at a decreasing rate until some satiation level is reached. In the latter stage, increase in prey consumption takes place slowly. There are several forms of functional response. Much works has been devoted to comparing two idealized forms of the functional response: prey dependent and ratio dependent. Recently, many biologists argue that in many situations the ratio-dependent theory is more suitable (see, for example, [1,15,21]). Ratio-dependence assumes that the functional response is a function of the ratio of prey over predators. The ratio-dependent predator–prey system was proposed by Ardini and Ginzburg [4] and then was widely studied in [7,11,25]. The ratio-dependent predator–prey system with diffusion was explored in [17].

Biological species can undergo discrete changes of relatively short duration at fixed times (for examples, due to stocking or harvesting of species). Moreover, continuous changes in environmental parameters such as temperature or rainfall can also create discontinuous outbreaks in pest populations. Systems with such kinds of discontinuous changes can be investigated by the theory of impulsive differential equations (see [3,2,12,13,19,6] and papers cited there). Therefore, there have been a number of studies which have applied the theory to biological problems (see, for example, [5,26,23,10]).

We consider an interaction of predator and prey species, assuming that they are confined to a fixed bounded space domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, non-uniformly distributed in the domain and subjected to short-term external influence at fixed moments of time. The functions $u(t,x)$ and $v(t,x)$ determine the densities of prey and predator, respectively, at a space point x and time t. Denote by $\partial/\partial n$ the outward normal derivative, $\bar{\Omega} = \Omega \cup \partial\Omega$, and $\Delta u = \partial^2 u/\partial x_1^2 + \cdots + \partial^2 u/\partial x_n^2$ the Laplace operator. Let also $\{t_k\}$ be a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_k < \cdots$.

The main object of this investigation is the following two-dimensional impulsive predator–prey system:

$$\frac{\partial u}{\partial t} = \mu_1 \Delta u + u\left(a_1(t,x) - b(t,x)u - \frac{c_1(t,x)v}{r(t,x)v+u}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} = \mu_2 \Delta v + v\left(-a_2(t,x) + \frac{c_2(t,x)u}{r(t,x)v+u}\right), \tag{2}$$

$$u(t_k+0,x) = u(t_k,x)f_k(x,u(t_k,x),v(t_k,x)), \tag{3}$$

$$v(t_k+0,x) = v(t_k,x)g_k(x,u(t_k,x),v(t_k,x)), \quad k = 1,2,\ldots, \tag{4}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{5}$$

In (1) and (2), terms $\mu_1\Delta u$ and $\mu_2\Delta v$ with positive diffusion coefficients μ_1 and μ_2 reflect the non-homogeneous dispersion of populations. Neumann boundary conditions (5) characterize the absence of migration. In the absence of predator, the prey population has a logistic growth rate. We assume that the predator functional response has the form of the ratio function $c_1 v/(mv+u)$. The ratio function $c_2 u/(mv+u)$ represents the conversion of prey to predator. a_1, c_1, a_2, and c_2 are positive functions that stand for prey intrinsic growth rate, capturing rate of the predator, death rate of the predator and conversion rate, respectively, a_1/b gives the carrying capacity of the prey, and r is the half saturation function.

In this article, we will investigate the asymptotic behavior of non-negative solutions for impulsive system with diffusion (1)–(5). Note that according to biological interpretation of the solutions $u(t,x)$ and $v(t,x)$ they must be nonnegative. We will give conditions for the long-term survival of each species in terms of permanence. The permanence of the system indicates that the number of individuals of each species stabilizes in certain boundaries with time. Note that impulsive system with another form of functional response (for example, prey-dependent functional response $u/(m+v)$) also can be studied by our approach. But conditions of boundedness and permanence and their proofs are essentially different from ratio-dependent case. This will be a subject of separate research.

The article is organized as follows. Main definitions and auxiliary results are provided in Section 2. In Section 3, conditions for the ultimate boundedness of solutions and permanence of the system are considered. The main subject of Section 4 is the compactness of bounded solutions. In the last section, we establish conditions for the existence of the unique periodic solution of the system.

2. Preliminaries

Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ be the sets of all integers, positive integers and real numbers, respectively, and $\mathbb{R}_+ = [0, \infty)$. The following assumptions will be needed throughout the paper:

(C1) functions $a_i(t, x), c_i(t, x), i = 1, 2,\ b(t, x)$ and $r(t, x)$ are bounded positive-valued functions on $\mathbb{R} \times \bar{\Omega}$, continuously differentiable in t and x;
(C2) functions $f_k(x, u, v)$ and $g_k(x, u, v), k \in \mathbb{N}$, are continuously differentiable in all arguments and positive-valued;
(C3) functions $a_i(t, x), c_i(t, x), i = 1, 2,\ b(t, x)$ and $r(t, x)$ are periodic in t with a period $\omega > 0$;
(C4) there exists a number $p \in \mathbb{N}$ such that $t_{k+p} = t_k + \omega$ for all $k \geqslant 1$;
(C5) sequences f_k and g_k satisfy the following equalities $f_{k+p}(x, u, v) = f_k(x, u, v)$ and $g_{k+p}(x, u, v) = g_k(x, u, v)$ for all $k \geqslant 1$ and x, u, v.

Conditions of periodicity (C3)–(C5) are natural because of the seasonal changes and biological rhythms.

We introduce the following notations: $G = \mathbb{R}_+ \times \Omega$, $\bar{G} = \mathbb{R}_+ \times \bar{\Omega}$,

$$\Gamma_k = \{(t, x) : t \in (t_{k-1}, t_k),\ x \in \Omega\}, \quad k \in \mathbb{N}, \quad \Gamma = \bigcup_{k \in \mathbb{N}} \Gamma_k,$$

$$\bar{\Gamma}_k = \{(t, x) : t \in (t_{k-1}, t_k),\ x \in \bar{\Omega}\}, \quad k \in \mathbb{N}, \quad \bar{\Gamma} = \bigcup_{k \in \mathbb{N}} \bar{\Gamma}_k$$

and denote by Ξ a class of functions $\phi : \bar{G} \to \mathbb{R}$ with the following properties:

(S1) $\phi(t, x) \in C^{1,2}_{t,x}(\Gamma_k),\ \phi(t, x) \in C^{1,1}_{t,x}(\bar{\Gamma}_k),\ k \in \mathbb{N}$;
(S2) for all $k \in \mathbb{N},\ x \in \Omega$, there exist the following limits:

$$\lim_{s \to t_k - 0} \phi(s, x) = \phi(t_k, x), \qquad \lim_{s \to t_k + 0} \phi(s, x) = \phi(t_k + 0, x).$$

We shall call a vector-function $(u(t, x), v(t, x)) \in \Xi \times \Xi$ a solution of Problems (1)–(5) if it satisfies (1), (2) on Γ, (5) by $x \in \partial\Omega$, and (3), (4) for every $k \in \mathbb{N}$.

For a bounded function $\phi(t, x)$, we denote $\phi^L = \inf_{(t,x)} \phi(t, x),\ \ \phi^M = \sup_{(t,x)} \phi(t, x)$.

Consider the following logistic differential equation with impulsive actions:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = az(b - z), \quad t \neq t_k,$$

$$z(t_k + 0) = z(t_k)\lambda_k(z(t_k)), \quad k \in \mathbb{N}, \tag{6}$$

where $z \in \mathbb{R}_+$, a and b are positive constants, strictly increasing sequence $\{t_k\}$ satisfies condition (C4), and $\lambda_k, k \in \mathbb{N}$, are continuous positive-valued functions such that $\lambda_{k+p}(z) = \lambda_k(z)$ for all $z \in \mathbb{R}_+, k \in \mathbb{N}$. Condition (C4) implies that $t_{k+1} - t_k \geqslant \theta = \min_{i=0,1,\ldots,p}(t_{i+1} - t_i), k \geqslant 1$. Denote $A = b/(1 - \mathrm{e}^{-ab\theta})$, $B = \max_{k=1,\ldots,p}\max_{z\in[0,A]}\lambda_k(z)$, $C = \max(A, B)$.

The following assertion is significant as an auxiliary result of our paper.

Lemma 1. *Every solution* $z(t) = z(t, 0, z_0)$, $z_0 > 0$, *of* (6) *satisfies* $0 < z(t) \leqslant C$, *if* $t \geqslant \theta$.

Proof. For $t \in (0, t_1]$, we have that

$$z(t) = \frac{bz_0}{z_0(1 - \mathrm{e}^{-abt}) + b\,\mathrm{e}^{-abt}}. \tag{7}$$

It is obvious that the solution is positive-valued on the interval. Moreover, if $\theta \leqslant t \leqslant t_1$, then

$$z(t) \leqslant \frac{bz_0}{z_0(1-\mathrm{e}^{-abt}) + b\mathrm{e}^{-abt}} \leqslant \frac{b}{1-\mathrm{e}^{-ab\theta}} = A. \tag{8}$$

Particularly, $0 < z(t_1) \leqslant A$, and, hence, $0 < z(t_1+0) = z(t_1)\lambda_1(z(t_1)) < B$. It is easy to show that $0 < z(t) \leqslant \max(A, B) = C$ if $t \in [t_1, t_2]$, and, similarly to (8) one can verify that $0 < z(t_2) \leqslant A$. Further, in the same manner we can show that $0 < z(t) \leqslant C$ if $t \in (t_k, t_{k+1}]$, $k = 2, 3, \ldots$. □

The following comparison theorems will be needed throughout the paper.

Theorem 2 (Walter [24]). *Suppose that vector-functions $v(t,x) = (v_1(t,x), \ldots, v_m(t,x))$ and $w(t,x) = (w_1(t,x), \ldots, w_m(t,x))$, $m \geqslant 1$, satisfy the following conditions*:

(i) *they are of class C^2 in x, $x \in \Omega$ and of class C^1 in $(t,x) \in [a,b] \times \bar{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary*;

(ii) $v_t - \mu\Delta v - g(t,x,v) \leqslant w_t - \mu\Delta w - g(t,x,w)$, *where* $(t,x) \in [a,b] \times \Omega$, $\mu = (\mu_1, \ldots, \mu_m) > 0$ *(inequalities between vectors are satisfied coordinate-wise), vector-function $g(t,x,u) = (g_1(t,x,u), \ldots, g_m(t,x,u))$ is continuously differentiable and quasi-monotonically increasing with respect to $u = (u_1, \ldots, u_m)$*:

$$\frac{\partial g_i(t,x,u_1,\ldots,u_m)}{\partial u_j} \geqslant 0, \quad i,j = 1,\ldots,m, \quad i \neq j;$$

(iii) $\partial v/\partial n = \partial w/\partial n = 0$, $(t,x) \in [a,b] \times \partial\Omega$;
Then $v(t,x) \leqslant w(t,x)$ *for* $(t,x) \in [a,b] \times \bar{\Omega}$.

Theorem 3 (Smith [22]). *Assume that T and d are positive real numbers, a function $u(t,x)$ is continuous on* $[0,T] \times \bar{\Omega}$, *continuously differentiable in $x \in \bar{\Omega}$, with continuous derivatives $\partial^2 u/\partial x_i \partial x_j$ and $\partial u/\partial t$ on* $(0,T] \times \Omega$, *and $u(t,x)$ satisfies the following inequalities*:

$$\frac{\partial u}{\partial t} - d\Delta u + c(t,x)u \geqslant 0, \quad (t,x) \in (0,T] \times \Omega,$$

$$\frac{\partial u}{\partial n} \geqslant 0, \quad (t,x) \in (0,T] \times \partial\Omega,$$

$$u(0,x) \geqslant 0, \quad x \in \Omega,$$

where $c(t,x)$ is bounded on $(0,T] \times \Omega$. *Then $u(t,x) \geqslant 0$ on* $(0,T] \times \bar{\Omega}$.
Moreover, $u(t,x)$ is strictly positive on $(0,T] \times \bar{\Omega}$ *if $u(t,x)$ is not identically zero.*

After changes in variables $u \to u$, $v \to -v$, the system without impulses (1), (2), (5) satisfies conditions of Theorem 2. Using the method of upper and lower solutions for quasi-monotone systems (see [18]), we can verify that, for continuously differentiable initial functions $u_0(x) : \bar{\Omega} \to \mathbb{R}_+$, $v_0(x) : \bar{\Omega} \to \mathbb{R}_+$, $u_0(x) \not\equiv 0$, $u_0(x) \not\equiv 0$, there exists a classical solution of system (1), (2), (5), which can be extended to the semi-axis $t > 0$. A vector-function $(u(t,x), v(t,x))$ is the classical solution of system without impulses (1), (2), (5), if it is of class C^2 in x, $x \in \Omega$, of class C^1 in x, $x \in \bar{\Omega}$, of class C^1 in t, $t > 0$, and satisfies the system.

Using the existence of solutions of system (1), (2), (5) we can verify the existence of solutions for impulsive system (1)–(5). Indeed, if $0 < t \leqslant t_1$, the solutions of the system are well-defined as classical solutions of system without impulses (1), (2), (5). Impulsive conditions (3) and (4) imply that the functions $(u(t_1+0,x), v(t_1+0,x))$ are continuously differentiable in x, and satisfy boundary conditions (5). Hence, assuming $(u(t_1+0,x), v(t_1+0,x))$ as a new initial functions we can continue the solution on $(t_1, t_2]$. Proceeding in this way, we can construct the solution for all $t > 0$.

Since we consider, according to biological interpretation, only non-negative solutions of the system, the following assertion is of major importance.

Lemma 4. *Assume that conditions* (C1)–(C5) *hold. Then non-negative and positive quadrants of* $\mathbb{R}^2$ *are positively invariant for Problems* (1)–(5).

Proof. Let us consider only the first coordinate $u(t,x)$ of a solution since the proof for the second one is very similar. It can be simply verified that $\hat{u}(t,x)$ and $\tilde{u}(t,x)$ such that

$$\frac{\partial \hat{u}}{\partial t} - \mu_1 \Delta \hat{u} - \hat{u}\left(a_1^L - b^M \hat{u} - \frac{c_1^M}{r^L}\right) = 0, \quad \hat{u}(0,x) = u_0(x),$$

$$\frac{\partial \tilde{u}}{\partial t} - \mu_1 \Delta \tilde{u} - \tilde{u}(a_1^M - b^L \tilde{u}) = 0, \quad \tilde{u}(0,x) = u_0(x),$$

are lower and upper solutions of Eq. (1). Then, since $u_0(x) \geqslant 0, u_0(x) \not\equiv 0$, by Theorem 3, we get $\hat{u}(t,x) > 0$ and $\tilde{u}(t,x) > 0$ for $t \in (0, t_1]$. Since $u(t,x)$ is bounded from below by positive function $\hat{u}(t,x)$, we have $u(t,x) > 0$ for $t \in (0, t_1]$. Taking into account positiveness of the function f_1, we can repeat the same argument to prove the positiveness of $u(t,x)$ for $t \in [t_1, t_2]$. By induction, we have that $u(t,x) > 0$ for $t \in (0, \infty)$. □

3. Permanence

Definition 1. Solutions of system (1)–(5) are said to be ultimately bounded if there exist positive constants N_1 and N_2 such that for every solution $(u(t,x,u_0,v_0), v(t,x,u_0,v_0))$ there exists a moment of time $\bar{t} = \bar{t}(u_0,v_0) > 0$ such that $u(t,x,u_0,v_0)) \leqslant N_1$, $v(t,x,u_0,v_0)) \leqslant N_2$ for all $x \in \bar{\Omega}$, and $t \geqslant \bar{t}$.

Definition 2. System (1)–(5) is called permanent if there exist positive constants m_1, m_2, N_1 and N_2 such that for every solution with non-negative initial functions $u_0(x)$ and $v_0(x)$, $u_0(x) \not\equiv 0$, $v_0(x) \not\equiv 0$, there exists a moment of time $\tilde{t} = \tilde{t}(u_0,v_0)$ such that

$$m_1 \leqslant u(t,x,u_0,v_0)) \leqslant N_1, \quad m_2 \leqslant v(t,x,u_0,v_0)) \leqslant N_2$$

for all $x \in \bar{\Omega}$, and $t \geqslant \tilde{t}$.

Theorem 5. *Suppose that conditions* (C1)–(C5) *hold, and, moreover*:

(i) *there exists a positive-valued function* $\gamma(M)$ *such that* $f_k(x,u,v) \leqslant \gamma(M)$ *if* $k \in \mathbb{N}$, $u \leqslant M$, $v \geqslant 0$, *and* $x \in \bar{\Omega}$;
(ii) *the inequality*

$$-\omega a_2^L + \sum_{i=1}^{p} \ln g_i < 0, \tag{9}$$

holds, where $g_i = \sup_{(x,u,v)} g_i(x,u,v)$.

Then all solutions of system (1)–(5) *with non-negative initial functions are ultimately bounded*.

Proof. Let $\bar{u}(t,x,u_0)$ be a solution of the equation

$$\frac{\partial \bar{u}}{\partial t} - \mu_1 \Delta \bar{u} - \bar{u}(a_1^M - b^L \bar{u}) = 0. \tag{10}$$

Using inequality

$$0=\frac{\partial u}{\partial t}-\mu_1\Delta u-u\left(a_1(t,x)-b(t,x)u-\frac{c_1(t,x)v}{r(t,x)v+u}\right)\geqslant\frac{\partial u}{\partial t}-\mu_1\Delta u-u(a_1^M-b^Lu)$$

we obtain

$$0=\frac{\partial\bar u}{\partial t}-\mu_1\Delta\bar u-\bar u(a_1^M-b^L\bar u)\geqslant\frac{\partial u}{\partial t}-\mu_1\Delta u-u(a_1^M-b^Lu).$$

Applying Theorem 3, we conclude that $u(t,x,u_0,v_0)\leqslant\bar u(t,M_u)$, where constant M_u is such that $\|u_0(x)\|_C=\max_{x\in\bar\Omega}|u_0(x)|\leqslant M_u$. Note that, according to the uniqueness theorem, the solution $\bar u(t,M_u)$ of Eq. (10) with initial condition independent of x does not depend on x for $t>0$. Therefore, the function $\bar u(t,M_u)$ satisfies the ordinary differential equation $\mathrm{d}\bar u/\mathrm{d}t=\bar u(a_1^M-b^L\bar u)$. Hence,

$$\begin{aligned}\|u(t_k+0,x,u_0,v_0)\|_C&=\|u(t_k,x,u_0,v_0)f_k(u(t_k,x,u_0,v_0),v(t_k,x,u_0,v_0))\|_C\\&\leqslant\bar u(t_k,M_u)\gamma(\bar u(t_k,M_u)).\end{aligned}$$

Since all solutions of the impulsive differential equation

$$\frac{\mathrm{d}\bar u}{\mathrm{d}t}=\bar u(a_1^M-b^L\bar u),\quad \bar u(t_k+0)=\bar u(t_k)\gamma(\bar u(t_k))$$

are ultimately bounded by Lemma 3, we get ultimately boundedness of solutions of Eq. (1) with impulses (3), i.e. there exists a positive constant N_1 such that $u(t,x)\leqslant N_1$, starting with some moment of time.

For the predator equation, from inequalities

$$0=\frac{\partial v}{\partial t}-\mu_2\Delta v+a_2(t,x)v-\frac{c_2(t,x)uv}{r(t,x)v+u}\geqslant\frac{\partial v}{\partial t}-\mu_2\Delta v+a_2^Lv-\frac{c_2^MN_1}{r^L},$$

it follows that $v(t,x,u_0,v_0)\leqslant\bar v(t,M_v)$, where $\bar v(t,M_v)$ is a solution of the initial value problem $\mathrm{d}\bar v/\mathrm{d}t=-a_2^L\bar v+c_2^MN_1/r^L,\ \bar v(0,M_v)=M_v$.

Linear periodic impulsive equation

$$\frac{\mathrm{d}\bar v}{\mathrm{d}t}=-a_2^L\bar v+\frac{c_2^MN_1}{r^L},\quad \bar v(t_k+0)=g_k\bar v(t_k) \tag{11}$$

has the general solution $\bar v(t)=X_0(t)+CX(t)$, where $X_0(t)$ is a ω-periodic piece-wise continuous function, C is a constant and

$$X(t)=\exp\left(-a_2^Lt+\sum_{0<t_k<t}\ln g_k\right)$$

(see [19]). By (9), $X(t)\to0$ as $t\to\infty$. All solutions of (11) are ultimately bounded, therefore, all solutions of Eqs. (2) and (4) are ultimately bounded, too. □

Theorem 6. *Suppose that conditions* (C1)–(C5) *hold and*, *moreover*:

(i) *solutions are ultimately bounded*, *i.e.*, *there exist positive constants* N_1 *and* N_2 *such that for every solution* $(u(t,x,u_0,v_0),v(t,x,u_0,v_0))$ *there exists* $\bar t=\bar t(u_0,v_0)>0$ *such that* $u(t,x,u_0,v_0)\leqslant N_1$ *and* $v(t,x,u_0,v_0)\leqslant N_2$ *for all* $t\geqslant\bar t(u_0,v_0)$;

(ii) *the following inequalities hold*:

$$\sum_{i=1}^{p} \ln \inf_{x\in\Omega,(u,v)\in S} f_i(x,u,v) + \omega\left(a_1^L - \frac{c_1^M}{r^L}\right) > 0, \tag{12}$$

$$\sum_{i=1}^{p} \ln \inf_{x\in\Omega,(u,v)\in S} g_i(x,u,v) + \omega(c_2^L - a_2^M) > 0, \tag{13}$$

where $S=\{(u,v): 0<u\leqslant N_1, 0<v\leqslant N_2\}$.

Then there exist positive constants α_1^* *and* α_2^* *such that an arbitrary solution of system* (1)–(5) *with non-negative initial functions not identically equal to zero satisfies the condition*

$$(u(t,x), v(t,x)) \in E = \{(u,v): \alpha_1^* \leqslant u(t,x) \leqslant N_1,\ \alpha_2^* \leqslant v(t,x) \leqslant N_2\},$$

starting with a certain moment of time.

Proof. Theorem 3 implies that if $u_0(x)\geqslant 0$, $u_0(x)\not\equiv 0$, and $v_0(x)\geqslant 0$, $v_0(x)\not\equiv 0$, then $u(t,x,u_0,v_0)>0$, $v(t,x,u_0,v_0)>0$ for all $x\in\bar{\Omega}$ and $t>0$. Considering the solution on the interval $t\geqslant\varepsilon$ with some small $\varepsilon>0$, we get initial conditions $(u(\varepsilon,x,u_0,v_0), v(\varepsilon,x,u_0,v_0))$ separated from zero. Therefore, we can assume, without loss of generality, that $\min_{x\in\bar{\Omega}} u_0(x)=m_u>0$ and $\min_{x\in\bar{\Omega}} v_0(x)=m_v>0$.

Using the inequality

$$0=\frac{\partial u}{\partial t}-\mu_1\Delta u - u\left(a_1(t,x)-b(t,x)u-\frac{c_1(t,x)v}{r(t,x)v+u}\right) \leqslant \frac{\partial u}{\partial t}-\mu_1\Delta u - u\left(a_1^L - b^M u - \frac{c_1^M}{r^L}\right),$$

we obtain that

$$0=\frac{\partial \hat{u}}{\partial t}-\mu_1\Delta \hat{u} - \hat{u}\left(a_1^L - b^M \hat{u} - \frac{c_1^M}{r^L}\right) \leqslant \frac{\partial u}{\partial t}-\mu_1\Delta u - u\left(a_1^L - b^M u - \frac{c_1^M}{r^L}\right).$$

Now, using Theorem 2 for $m=1$, we have that $u(t,x,u_0,v_0)\geqslant\hat{u}(t,m_u)$ for $t\in[0,t_1]$. Applying the last inequality for $t=t_1$, and (3) we obtain that

$$u(t_1+0,x,u_0,v_0)\geqslant\hat{u}(t_1,m_u)\inf_{x\in\Omega,(u,v)\in S} f_1(x,u,v).$$

Thus, the solution $u(t,x,u_0,v_0)$ is bounded from below by a solution of periodic logistic equation with impulses

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t}=\hat{u}\left(a_1^L-\frac{c_1^M}{r^L}-b^M\hat{u}\right),\quad \hat{u}(t_i+0)=\hat{u}(t_i)\inf_{x\in\Omega,(u,v)\in S} f_i(x,u,v). \tag{14}$$

By Theorem 2.1 [14] and condition (12), Eq. (14) has a unique piece-wise continuous and strictly positive periodic solution $\hat{u}^*(t)$ such that every solution $\hat{u}(t,u_m)$ of (14) with $u_m>0$ has the property $\hat{u}(t,u_m)\to\hat{u}^*(t)$ as $t\to\infty$. Therefore, there exists a positive constant α_1^* such that, for every solution $\hat{u}(t,u_m)$, $u_m>0$, of Eq. (14) we get $\hat{u}(t,u_m)\geqslant\alpha_1^*$, starting with some moment of time $\hat{t}=\hat{t}(u_m)>0$.

Since solution $u(t,x,u_0,v_0)$ of Eqs. (1), (3) is bounded from below by solution $\hat{u}(t,u_m)$ of Eq. (14), we conclude that $u(t,x,u_0,v_0)\geqslant\alpha_1^*$ for $t\geqslant\hat{t}$.

Let us consider the predator equation. Since $u(t,x,u_0,v_0)\geqslant\alpha_1^*$, we have

$$0=\frac{\partial v}{\partial t}-\mu_2\Delta v + a_2(t,x)v - \frac{c_2(t,x)uv}{r(t,x)v+u} \leqslant \frac{\partial v}{\partial t}-\mu_2\Delta v + (a_2^M - c_2^L)v + \frac{c_2^L r^M v^2}{r^M v + \alpha_1^*}.$$

Hence, $v(t,x,u_0,v_0) \geqslant \hat{v}(t,m_v)$, where $\hat{v}(t,m_v)$, $\hat{v}(0,m_v)=m_v$, is the solution of equation

$$\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = (c_2^L - a_2^M)\hat{v} - \frac{c_2^L r^M \hat{v}^2}{r^M \hat{v} + \alpha_1^*}, \quad \hat{v}(t_i+0) = \hat{v}(t_i)\hat{g}_i, \tag{15}$$

where $\hat{g}_i = \inf_{x\in\Omega,(u,v)\in S} g_i(x,u,v)$. If $\hat{v}(t) \leqslant \alpha_2$ for $t \in [0,t_1]$, then

$$\hat{v}(t_1,m_v) \geqslant m_v \exp\left(t_1\left(c_2^L - a_2^M - \frac{c_2^L r^M \alpha_2}{r^M\alpha_2 + \alpha_1^*}\right)\right)$$

and

$$\hat{v}(t_1+0,m_v) \geqslant \hat{g}_1 m_v \exp\left(t_1\left(c_2^L - a_2^M - \frac{c_2^L r^M \alpha_2}{r^M\alpha_2 + \alpha_1^*}\right)\right).$$

Therefore, if $\hat{v}(t) \leqslant \alpha_2$ for $t \in [0,\omega]$ then

$$\hat{v}(\omega,m_v) \geqslant m_v \exp\left(\sum_{i=1}^{p} \ln \hat{g}_i + \omega\left(c_2^L - a_2^M - \frac{c_2^L r^M \alpha_2}{r^M\alpha_2 + \alpha_1^*}\right)\right).$$

Taking into account (13), we can take sufficiently small $\alpha_2 > 0$ such that

$$\sum_{i=1}^{p} \ln \hat{g}_i + \omega\left(c_2^L - a_2^M - \frac{c_2^L r^M \alpha_2}{r^M\alpha_2 + \alpha_1^*}\right) = h_2 > 0.$$

For $\alpha_2^0 \in (0,\alpha_2)$, there exists a positive integer k_2 such that $\hat{v}(k_2\omega,m_v) \geqslant \mathrm{e}^{h_2 k_2} m_v \geqslant \alpha_2^0$ (by the additional condition $\hat{v}(t,m_v) < \alpha_2$ for all $t \in [0,k_2\omega]$).

Hence, for every solution $\hat{v}(t,\hat{v}_0)$ of (15) with $\hat{v}_0 > 0$ there exists a moment of time $\hat{t}$ such that $\hat{v}(\hat{t},\hat{v}_0) \geqslant \alpha_2^0$. Denote by $\hat{v}(t,\tau,\hat{v}_0)$ the solution of (15) with $\hat{v}(\tau,\tau,\hat{v}_0) = \hat{v}_0$ and consider a positive number

$$\alpha_2^* = \inf\{\hat{v}(t,\tau,\hat{v}_0) : \tau \in [0,\omega], \hat{v}_0 \in [\alpha_2^0, N_2], t \in [\tau, 2\omega]\}.$$

Then $\hat{v}(t,\tau,\hat{v}_0) \geqslant \alpha_2^*$ for all $t \geqslant 2\omega$. Indeed, let us take

$$\alpha_\omega = \inf\{\hat{v}(\omega,\tau,\hat{v}_0) : \tau \in [0,\omega], \hat{v}_0 \in [\alpha_2^0, N_2]\} \geqslant \alpha_2^*$$

and consider a solution $\hat{v}(t,\omega,\hat{v}_0)$ with $\hat{v}_0 \geqslant \alpha_\omega$. If $\hat{v}(t,\omega,\hat{v}_0) \leqslant \alpha_2$ for all $t \in [t,2\omega]$, then $\hat{v}(2\omega,\omega,\hat{v}_0) \geqslant h_2 \hat{v}(\omega,\omega,\hat{v}_0) \geqslant \alpha_\omega$. If $\hat{v}(t,\omega,\hat{v}_0) > \alpha_2$ at some moment of time $t \in [\omega,2\omega]$, then $\hat{v}(2\omega,\omega,\hat{v}_0) \geqslant \alpha_\omega$ by definition of number α_ω. Therefore, it is enough to consider solutions $\hat{v}(t,2\omega,\hat{v}_0)$, $t \geqslant 2\omega$, with $\hat{v}_0 \geqslant \alpha_\omega$. By construction, these solutions are bounded from below by positive constant α_2^* for $t \in [2\omega,3\omega]$. Proceeding in this way we prove the boundedness from below for $t \geqslant 3\omega$. □

Remark 7. If there are not any impulses in Problems (1)–(5) ($f_k \equiv 1$, $g_k \equiv 1$, $k \geqslant 1$), then conditions (12), (13) take a form of the following inequalities:

$$\left(a_1^L - \frac{c_1^M}{r^L}\right) > 0, \quad (c_2^L - a_2^M) > 0, \tag{16}$$

which are sufficient to have the permanence phenomenon for the system.

Assume that inequalities (16) are violated. Nevertheless, choosing functions f_k, g_k positive and separated sufficiently far away from zero we can satisfy (12), (13), that is we can arrange the permanence for the impulsive system again. This method of "improving" of differential equations by involving appropriate impulses are very useful for theoretical as well as for practical needs [12,19].

The next theorem gives conditions for the extinction of the predator population.

Theorem 8. *Suppose that system* (1)–(5) *satisfies conditions* (C1)–(C5), *and*, *moreover*,

$$\sum_{i=1}^{p} \ln \sup_{(x,u,v)} g_i(x,u,v) + \omega(c_2^M - a_2^L) < 0. \tag{17}$$

Then $v(t,x) \to 0$ *as* $t \to \infty$.

Proof. Fix a positive constant M_v such that $M_v \geqslant v_0(x)$ and denote by $\bar{v}(t, M_v)$ the solution of initial value problem

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = (c_2^M - a_2^L)\bar{v}, \quad \bar{v}(0, M_v) = M_v.$$

From the inequality

$$0 = \frac{\partial v}{\partial t} - \mu_2 \Delta v + a_2(t,x)v - \frac{c_2(t,x)uv}{r(t,x)v+u} \geqslant \frac{\partial v}{\partial t} - \mu_2 \Delta v + (a_2^L - c_2^M)v,$$

applying the comparison theorem again, we can find that $v(t,x,u_0,v_0) \leqslant \bar{v}(t, M_v)$ for $t \leqslant t_1$.

Moreover, using impulsive condition (4) we obtain that

$$v(t_1+0, x, u_0, v_0) \leqslant \bar{v}(t_1, M_v) \sup_{(x,u,v)} g_1(x,u,v).$$

Proceeding in this fashion, we conclude that every solution of the predator equation with impulses is bounded from above by the corresponding solution of linear impulsive equation

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = (c_2^M - a_2^L)\bar{v}, \quad \bar{v}(t_k+0) = \bar{v}(t_k) \sup_{(x,u,v)} g_k(x,u,v).$$

Taking into account (17), we see that all solutions of the last equation tend to zero as $t \to \infty$. $\square$

4. Compactness of solutions

Denote $w = (u,v) \in L_p(\Omega) \times L_p(\Omega)$, where $p > n$ is a positive integer. We rewrite system (1)–(5) in the form

$$\frac{\mathrm{d}w}{\mathrm{d}t} = A_1 w + F(t,w), \quad t \neq t_i, \tag{18}$$

$$w(t_i+0) = w(t_i) + G_i(w(t_i)), \quad i \in \mathbb{N}, \tag{19}$$

where

$$A_1 = \begin{pmatrix} \mu_1 \Delta - \delta & 0 \\ 0 & \mu_2 \Delta - \delta \end{pmatrix}, \quad \delta > 0,$$

$$F(t,w) = \begin{pmatrix} u\left(a_1(t,x) - b(t,x)u - \dfrac{c_1(t,x)v}{r(t,x)v+u}\right) + \delta u \\ v\left(-a_2(t,x) + \dfrac{c_2(t,x)u}{r(t,x)v+u}\right) + \delta v \end{pmatrix},$$

$$G_i(w(t_i)) = \begin{pmatrix} u(t_i,x) f_i(x, u(t_i,x), v(t_i,x)) - u(t_i,x) \\ v(t_i,x) g_i(x, u(t_i,x), v(t_i,x)) - v(t_i,x) \end{pmatrix}.$$

The operator A_1 has the domain $\mathscr{D}(A_1)=\{\xi : \xi \in W^{2,p}(\Omega), \frac{\partial\xi}{\partial n}|_{\partial\Omega}=0\}$, where $W^{2,p}(\Omega)$ is the Sobolev space of functions from $L_p(\Omega)$ that have two generalized derivatives. It is known [8] that the operator A_1 is sectorial and $Re\,\sigma(A_1)\leqslant -\delta$, where $\sigma(A_1)$ is the spectrum of A_1. For any $\alpha>0$, we define the fractional power $A_1^{-\alpha}$ of operator A_1 by

$$A_1^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{-sA_1}s^{\alpha-1}\,\mathrm{d}s,$$

where Γ is the gamma function. The operators $A_1^{-\alpha}$ are bounded and bijective. The operator A_1^{α}, $\alpha>0$, is defined as $(A_1^{-\alpha})^{-1}$, and $\mathscr{D}(A_1^{\alpha})=\mathscr{R}(A_1^{-\alpha})$. The operator A_1^0 is the identity operator in X. For $0\leqslant\alpha\leqslant 1$, we introduce the space $X^\alpha=\mathscr{D}(A_1^\alpha)$ with the norm $\|x\|_\alpha=\|A_1^\alpha x\|$. Here $\|.\|$ is the norm in the space $X=L_p\times L_p$.

We denote by $C^{m+\alpha}(\Omega)$, where m is a positive integer and $0<\alpha<1$, the space of m-times continuously differentiable functions $f:\Omega\to\mathbb{R}$, which have m-order derivatives satisfying the Holder condition with exponent α.

Theorem 9. *Suppose that the functions G_i are continuously differentiable and there exists a positive-valued function $\eta(M)$ such that*

$$\sup_{\|w\|_\alpha\leqslant M}\|G_k(w)\|_\alpha\leqslant\eta(M),\quad k\in\mathbb{N},\tag{20}$$

for some $\alpha\in(\frac{1}{2}+\frac{n}{2p},1)$. Let $w(t,w_0)=(u(t,x,u_0,v_0),v(t,x,u_0,v_0))$, $w_0=(u_0,v_0)\in X^\alpha$, be a bounded solution of Eqs. (18), (19), *i.e.*,

$$\|w(t,w_0)\|_C\leqslant N,\quad t>0.\tag{21}$$

Then the set $\{w(t,w_0):t>0\}$ is relatively compact in $C^{1+v}(\bar{\Omega},\mathbb{R}^2)$ for $0<v<2\alpha-1-n/p$.

Proof. The solution of Eqs. (18), (19) can be written in the form

$$w(t,w_0)=\mathrm{e}^{A_1t}w_0+\int_0^t \mathrm{e}^{A_1(t-s)}F(s,w(s))\,\mathrm{d}s+\sum_{0<t_i<t}\mathrm{e}^{A_1(t-t_i)}G_i(w(t_i))\tag{22}$$

(see [19]). By definition of the norm in the space X^α, it follows from (22) that

$$\begin{aligned}\|w(t,w_0)\|_\alpha&=\|A_1^\alpha w(t,w_0)\|\\&\leqslant\|A_1^\alpha\mathrm{e}^{A_1t}w_0\|+\int_0^t\|A_1^\alpha\mathrm{e}^{A_1(t-s)}\|\|F(s,w(s))\|\,\mathrm{d}s+\sum_{0<t_i<t}\|A_1^\alpha\mathrm{e}^{A_1(t-t_i)}G_i(w(t_i))\|.\end{aligned}\tag{23}$$

It is known [8] that analytical semigroup e^{A_1t}, $t\geqslant 0$, satisfies the following inequality:

$$\|A_1^\alpha\mathrm{e}^{A_1t}\|\leqslant C_\alpha t^{-\alpha}\mathrm{e}^{-\delta t},\quad t>0.\tag{24}$$

Hence, one can verify that

$$\|A_1^\alpha\mathrm{e}^{A_1t}w_0\|\leqslant\|\mathrm{e}^{A_1t}\|\|A_1^\alpha w_0\|\leqslant C_0\mathrm{e}^{-\delta t}\|w_0\|_\alpha\leqslant C_0\|w_0\|_\alpha\tag{25}$$

and

$$\begin{aligned}\int_0^t\|A_1^\alpha\mathrm{e}^{A_1(t-s)}\|\|F(s,w(s))\|\,\mathrm{d}s&\leqslant\int_0^t C_\alpha(t-s)^{-\alpha}\mathrm{e}^{-\delta(t-s)}\,\mathrm{d}s\leqslant C_\alpha M_F\int_0^\infty s^{-\alpha}\mathrm{e}^{-\delta s}\,\mathrm{d}s\\&=C_\alpha M_F\frac{\Gamma(1-\alpha)}{\delta^{1-\alpha}},\end{aligned}\tag{26}$$

where $M_F=\sup_s\|F(s,w(s))\|<\infty$.

Let us fix $\varepsilon \in (0, \theta)$, where $\theta = \min_k (t_{k+1} - t_k)$, and consider the third term at the right-hand side of (23) for the values of t located to the right of the impulse points at a distance not less then ε. Then we have $(t - t_i)^{-\alpha} \leqslant \varepsilon^{-\alpha}$ and the following estimation is correct:

$$\sum_{0<t_i<t} \|A_1^\alpha e^{A_1(t-t_i)} G_i(w(t_i))\| \leqslant \sum_{0<t_i<t} C_\alpha (t-t_i)^{-\alpha} e^{-\delta(t-t_i)} \|G_i(w(t_i))\|$$
$$\leqslant C_\alpha M_G \varepsilon^{-\alpha} \sum_{j=0}^{\infty} e^{-\delta\theta j} \leqslant C_\alpha M_G \frac{\varepsilon^{-\alpha}}{1 - e^{-\delta\theta}}, \tag{27}$$

where $M_G = \sup_i \|G_i(w(t_i))\|$. Taking into account (21) and periodicity of G_j in j, we get $M_G < \infty$. Relations (25)–(27) yield $\|w(t, w_0)\|_\alpha \leqslant M_\varepsilon < \infty$ for $t \in \bigcup_i [t_i + \varepsilon, t_{i+1}]$, and, therefore, $\|w(t_k, w_0)\|_\alpha \leqslant M_\varepsilon$ for all k. By (20), we get $\|G_k(w_k)\|_\alpha \leqslant \eta(M_\varepsilon)$, $k \in \mathbb{N}$. Now, the sum in (23) can be estimated as follows:

$$\sum_{0<t_i<t} \|A_1^\alpha e^{A_1(t-t_i)} G_i(w(t_i))\| \leqslant \sum_{0<t_i<t} \|e^{A_1(t-t_i)}\| \|A_1^\alpha G_i(w(t_i))\| \leqslant \sum_{0<t_i<t} C_0 e^{-\delta(t-t_i)} \|G_i(w(t_i))\|_\alpha$$
$$\leqslant C_0 \eta(M_\varepsilon) \sum_{j=0}^{\infty} e^{-\delta\theta j} = \frac{C_0 \eta(M_\varepsilon)}{1 - e^{-\delta\theta}}. \tag{28}$$

This proves the boundedness of the solution $w(t, w_0)$ in the space X^α. By [8], the boundedness of a set in X^α implies its relative compactness in the space X^β for $\beta < \alpha$. For $2\beta - n/p > 1 + v$, the space X^β is continuously embedded in the space C^{1+v}. Therefore, the solution $w(t, w_0)$ is relatively compact in the space C^{1+v}. □

5. Periodic solutions

Theorem 10. *Assume that conditions* (C1)–(C5) *and* (20) *hold and system* (1)–(5) *is permanent, i.e., there exist positive constants β and N such that an arbitrary solution of the system with non-negative initial functions not identically equal to zero satisfies the condition*

$$(u(t, x), v(t, x)) \in \mathscr{E} = \{(u, v) : \beta \leqslant u(t, x) \leqslant N, \ \ \beta \leqslant v(t, x) \leqslant N\},$$

starting with a certain moment of time. Let, additionally,

$$\sum_{j=1}^{p} \ln K_j + \omega \lambda_M < 0,$$

where λ_M is the maximal eigenvalue of the matrix

$$\begin{pmatrix} 2\left(a_1^M - b^L\beta - \dfrac{c_1^L r^L}{(r^M + N/\beta)^2}\right) & c_1^M + \dfrac{c_2^M}{r^L} \\ c_1^M + \dfrac{c_2^M}{r^L} & 2\left(-a_2^L + \dfrac{c_2^M}{(1 + r^L\beta/N)^2}\right) \end{pmatrix}$$

and

$$K_j = \max_{u,v \in \mathscr{E}, x \in \Omega} 2\left(f_j^2 + \left(N\frac{\partial f_j}{\partial u}\right)^2 + \left(N\frac{\partial f_j}{\partial v}\right)^2 + g_j^2 + \left(N\frac{\partial g_j}{\partial u}\right)^2 + \left(N\frac{\partial g_j}{\partial v}\right)^2\right).$$

Then system (1)–(5) *has a unique globally asymptotically stable strictly positive piecewise continuous ω-periodic solution.*

Proof. Let $(u(t,x), v(t,x))$ and $(u_1(t,x), v_1(t,x))$ be two solutions of system (1)–(5) bounded by constants β and N from below and above, respectively. Consider the function

$$L(t) = \int_\Omega ((u(t,x) - u_1(t,x))^2 + (v(t,x) - v_1(t,x))^2)\,\mathrm{d}x.$$

Its derivative has the form

$$\begin{aligned}
\frac{\mathrm{d}L(t)}{\mathrm{d}t} &= 2\int_\Omega (u-u_1)\left(\frac{\partial u}{\partial t} - \frac{\partial u_1}{\partial t}\right)\mathrm{d}x + 2\int_\Omega (v-v_1)\left(\frac{\partial v}{\partial t} - \frac{\partial v_1}{\partial t}\right)\mathrm{d}x \\
&= 2\mu_1 \int_\Omega (u-u_1)\Delta(u-u_1)\,\mathrm{d}x \\
&\quad + 2\int_\Omega (u-u_1)\left(u\left(a_1 - bu - \frac{c_1 v}{rv+u}\right) - u_1\left(a_1 - bu_1 - \frac{c_1 v_1}{rv_1+u_1}\right)\right)\mathrm{d}x \\
&\quad + 2\mu_2 \int_\Omega (v-v_1)\Delta(v-v_1)\,\mathrm{d}x + 2\int_\Omega (v-v_1)\left(-a_2 v + \frac{c_2 uv}{rv+u} + a_2 v_1 - \frac{c_2 u_1 v_1}{rv_1+u_1}\right)\mathrm{d}x \\
&\leqslant -2\mu_1 \int_\Omega |\nabla(u-u_1)|^2\,\mathrm{d}x - 2\mu_2 \int_\Omega |\nabla(v-v_1)|^2\,\mathrm{d}x \\
&\quad + 2\int_\Omega (u-u_1)^2\left(a_1 - b(u+u_1) - \frac{c_1 r v v_1}{(rv+u)(rv_1+u_1)}\right)\mathrm{d}x \\
&\quad + 2\int_\Omega (v-v_1)^2\left(-a_2 + \frac{c_2 u u_1}{(rv+u)(rv_1+u_1)}\right)\mathrm{d}x \\
&\quad + 2\int_\Omega (u-u_1)(v-v_1)\left(\frac{-c_1 u u_1}{(rv+u)(rv_1+u_1)} + \frac{c_2 r v v_1}{(rv+u)(rv_1+u_1)}\right)\mathrm{d}x \\
&\leqslant 2\int_\Omega (u-u_1)^2\left(a_1^M - b^L\beta - \frac{c_1^L r^L}{(r^M + N/\beta)^2}\right)\mathrm{d}x \\
&\quad + 2\int_\Omega \left[(v-v_1)^2\left(-a_2^L + \frac{c_2^M}{(1 + r^L\beta/N)^2}\right) + |(u-u_1)(v-v_1)|\left(c_1^M + \frac{c_2^M}{r^L}\right)\right]\mathrm{d}x \\
&\leqslant \lambda_M \int_\Omega [(u-u_1)^2 + (v-v_1)^2]\,\mathrm{d}x.
\end{aligned}$$

Using the last inequality, we obtain $L(t_{j+1}) \leqslant L(t_j + 0)\exp(\lambda_M(t_{j+1} - t_j))$ and

$$\begin{aligned}
L(t_{j+1}+0) &= \int_\Omega (uf_{j+1}(u,v) - u_1 f_{j+1}(u_1,v_1))^2\,\mathrm{d}x + \int_\Omega (vg_{j+1}(u,v) - v_1 g_{j+1}(u_1,v_1))^2\,\mathrm{d}x \\
&\leqslant K_{j+1}L(t_{j+1}) \leqslant K_{j+1}\exp(\lambda_M(t_{j+1} - t_j))L(t_j + 0).
\end{aligned}$$

Let us estimate the variation of the function over the period. We have

$$L(t+\omega) \leqslant K_* L(t) = \prod_{i=1}^{p} K_i \exp(\lambda_M \omega) L(t).$$

According to the conditions of the theorem, we have $K_* < 1$. Therefore, $L(m\omega + s) \leqslant K_*^m L(s) \to 0,\ m \to \infty$. We have proved that $\|u(t,x) - u_1(t,x)\| \to 0$ and $\|v(t,x) - v_1(t,x)\| \to 0$ as $t \to \infty$, where $\|.\|$ is the norm of the space $L_2(\Omega)$. By Theorem 9, solutions of system (1)–(5) are bounded in the space $C^{1+\nu}$. Therefore

$$\sup_{x\in\Omega} |u(t,x) - u_1(t,x)| \to 0, \quad \sup_{x\in\Omega} |v(t,x) - v_1(t,x)| \to 0, \quad t \to \infty. \tag{29}$$

Now we consider the sequence $(u(k\omega, x, u_0, v_0), v(k\omega, x, u_0, v_0)) = w(k\omega, w_0)$, $k \in \mathbb{N}$. By Theorem 9, it is compact in the space $C(\bar\Omega) \times C(\bar\Omega)$. Let $\bar w$ be a limit point of this sequence, $\bar w = \lim_{n\to\infty} w(k_n\omega, w_0)$. Then $w(\omega, \bar w) = \bar w$.

Indeed, since $w(\omega, w(k_n\omega, w_0)) = w(k_n\omega, w(\omega, w_0))$ and $w(k_n\omega, w(\omega, w_0)) - w(k_n\omega, w_0) \to 0$ as $k_n \to \infty$, we get

$$\|w(\omega, \bar{w}) - \bar{w}\|_C \leqslant \|w(\omega, \bar{w}) - w(\omega, w(k_n\omega, w_0))\|_C + \|w(\omega, w(k_n\omega, w_0)) - w(k_n\omega, w_0)\|_C + \|w(k_n\omega, w_0) - \bar{w}\|_C \to 0, \quad n \to \infty.$$

The sequence $w(k\omega, w_0)$, $k \in \mathbb{N}$, has a unique limit point. On the contrary, let the sequence has two limit points $\bar{w} = \lim_{n\to\infty} w(k_n\omega, w_0)$ and $\hat{w} = \lim_{m\to\infty} w(k_m\omega, w_0)$. Then, taking into account (29) and $\hat{w} = w(k_n\omega, \hat{w})$, we have

$$\|\bar{w} - \hat{w}\|_C \leqslant \|\bar{w} - w(k_n\omega, w_0)\|_C + \|w(k_n\omega, w_0) - \hat{w}\|_C \to 0, \quad n \to \infty,$$

hence $\bar{w} = \hat{w}$. The solution $(u(t, x, \bar{u}, \bar{v}), v(t, x, \bar{u}, \bar{v}))$ is the unique periodic solution of system (1)–(5). By (29), it is asymptotically stable. □

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