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Perron's theorem for linear impulsive differential equations with distributed delay[☆]

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Abstract

In this paper it is shown that under a Perron condition trivial solution of linear impulsive differential equation with distributed delay is uniformly asymptotically stable.

Keywords: Perron condition; Stability; Adjoint; Impulse; Distributed delay

1. Introduction

Impulsive differential equations with delay may express the motion of some real world simulation processes which depend on their prehistory and are subject to short time disturbances. The prehistorical dependence may cause the presence of the delays through the differential equation as well as through the impulse conditions substantially affecting the motions. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnologies, economics, etc. [1,12,13,16–18,20]. For the theory of delay or impulsive differential equations, see the monographs [8–10,15].

In 1930, Perron [14] proved his celebrated theorem that if for every continuous function $f(t)$ bounded on $[0, \infty)$, the solution of the equation

$$x'(t) = A(t)x(t) + f(t), \quad x(0) = 0,$$

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is bounded on $[0, \infty)$, then the trivial solution of the corresponding homogeneous equation is uniformly asymptotically stable. Later, it was shown that the result of Perron is equivalent to having a fundamental matrix of the homogeneous equation which satisfies an exponential estimate. Indeed, a result in this direction was established earlier by Bohl in [6], where he proved that under the same hypothesis proposed by Perron, the fundamental matrix has the estimate $\|X(t)\| \leqslant N\mathrm{e}^{-\lambda t}$, where N and λ are positive constants. Obviously, this estimate yields the exponential stability of the trivial solution. Recently, the above results have been carried out for linear delay differential equations [3,7,19] and for linear impulsive differential equations [5].

In this paper, we consider a more general system of linear impulsive differential equations with distributed delay. Our system differs from the previous ones not only it is more general but also we allow delay terms in the impulse conditions. Such impulse conditions are more natural for delay differential equations.

2. Preliminaries

Let $\eta : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^{n\times n}$ be a kernel function, cf. [10], satisfying the following conditions:

(a) $\eta(t, s)$ is normalized so that $\eta(t, s) = 0$ for $s \in [0, \infty)$, $\eta(t, s) = \eta(t, -\tau)$ for $s \in (-\infty, -\tau]$;
(b) $\eta(t, s)$ is continuous in t on $[0, \infty)$ uniformly for $s \in [-\tau, 0]$;
(c) There exists a positive real number γ such that the total variation of $\eta(t, s)$ in s on $[-\tau, 0]$ for $t \geqslant 0$ is not larger than γ.

We shall consider the impulsive differential equations with distributed delay of the form

$$x'(t) = \int_{-\tau}^{0} \mathrm{d}_s\eta(t, s)x(t + s) + f(t), \quad t \neq \theta_i, \quad i \in \mathbb{N},$$
$$\Delta x(\theta_i) := x(\theta_i^+) - x(\theta_i) = A_{i0}x(\theta_i) + \sum_{-j \leqslant k<0} A_{ik}x(\theta_{i+k}) + \beta_i, \tag{1}$$

where $f \in C([0, \infty), \mathbb{R}^n)$, $x(\theta_i^+) := \lim_{t\to\theta_i^+} x(t)$, $A_{ik} \in \mathbb{R}^{n\times n}$, $\beta_i \in \mathbb{R}^n$ for $i \in \mathbb{N} := \{1, 2, 3, \ldots\}$.

By a solution of (1) on an interval J, we mean a function x defined on J such that x is continuous everywhere on J except possibly at $\theta_i \in J$ for $i \in \mathbb{N}$, where $x(\theta_i^+)$ and $x(\theta_i^-) := \lim_{t\to\theta_i^-} x(t)$ exist, $x(\theta_i) := x(\theta_i^-)$, and that x satisfies (1).

Throughout the paper we also assume that there exist positive real numbers ρ_k, $k=-j, -j+1, \ldots, -1$, and v such that

(d) $\|A_{ik}\| \leqslant \rho_k$ for all $i \in \mathbb{N}$;
(e) $\theta_i - \theta_{i-j} \leqslant v$ for all $i \in \mathbb{N}$.

Let $PLC([-\tau, 0], \mathbb{R}^n)$ denote the set of piecewise left continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ having a finite number of discontinuity points of the first kind. For given $t_0 \geqslant 0$ and $\phi \in PLC([-\tau, 0], \mathbb{R}^n)$, the initial value problem of (1) is to find a solution $x(t)$ of (1) such that

$$x(t + t_0) = \phi(t), \quad t \in [-\tau, 0]. \tag{2}$$

Under the above conditions, one can easily show that the initial value problem has a unique solution which belongs to the set $PLC([t_0-\tau,\infty),\mathbb{R}^n)$.

Our main interest in this paper is to prove that the trivial solution of homogeneous equation corresponding to (1), namely

$$\begin{aligned} x'(t) &= \int_{-\tau}^{0} \mathrm{d}_s\eta(t,s)x(t+s), \quad t\neq\theta_i \\ \Delta x(\theta_i) &= A_{i0}x(\theta_i) + \sum_{-j\leqslant k<0} A_{ik}x(\theta_{i+k}), \quad i\in\mathbb{N} \end{aligned} \tag{3}$$

is uniformly asymptotically stable under Perron's condition. We say that (3) verifies Perron's condition if

> *for every bounded* $f\in C([0,\infty),\mathbb{R}^n)$ *and every bounded sequence* $\{\beta_i\}$, *the solution* $x(t)$ *of* (1) *satisfying* (2) *with* $\phi(t)\equiv 0$ *is bounded for* $t\in[0,\infty)$.

In [2], Anokhin et al. considered as a special case of (1) linear impulsive delay differential equations of the form

$$\begin{aligned} &x'(t) + \sum_{i=1}^{m} A_i(t)x(h_i(t)) = r(t), \quad t\geqslant 0, \\ &x(\theta_i) = B_i x(\theta_i - 0), \quad i\in\mathbb{N}. \end{aligned} \tag{4}$$

and proved a Bohl type result. That is, they established an exponential estimate for the fundamental matrix and hence proved that the trivial solution of (4) is exponentially stable under Perron's condition. We remark that the impulse condition in (4) at $t=\theta_i$ is independent of the previous data, and it is natural to ask what happens if this is not the case. We will give an affirmative answer to this question, showing that the zero solution of impulsive delay equations of more general type is uniformly asymptotically stable under Perron's condition. Our approach is based on the scheme proposed by Halanay in [8] and is completely different from the one used in [2].

The main result of this paper is the following theorem.

Theorem 1. *If* (3) *verifies Perron condition then its trivial solution is uniformly asymptotically stable.*

3. Adjoint equation and representation of solutions

In this section we establish a representation formula for solutions of (1) to be used in the proof of Theorem 1. For our purpose we first need to construct the adjoint of (3). Suppose that $\det(I+A_{i0})\neq 0$, and that there exists positive real number $\bar\rho_0$ such that $\|(I+A_{i0})^{-1}\|\leqslant\bar\rho_0$ for all $i\in\mathbb{N}$.

Let $x(t)$ be any solution of (1) and $y(t)$ any solution of

$$\begin{aligned} y'(t) &= -\mathrm{d}_t\int_{-\tau}^{0} \eta^T(t-s,s)y(t-s)\,\mathrm{d}s, \quad t\neq\theta_i, \\ \Delta y(\theta_i) &= -(I+A_{i0}^T)^{-1}\left[A_{i0}^T y(\theta_i) + \sum_{-j\leqslant k<0} A_{(i-k)k}^T y(\theta_{i-k}^+)\right]. \end{aligned} \tag{5}$$

It is not difficult to see that the integration of $x'^T(\alpha)y(\alpha)$ over an interval $[\sigma, t]$ leads to

$$\begin{aligned}x^T(t)y(t)-x^T(\sigma)y(\sigma)={}&\int_\sigma^t\left[\int_{-\tau}^0 x^T(\alpha+s)\,\mathrm{d}_s\eta^T(\alpha,s)+f^T(\alpha)\right]y(\alpha)\,\mathrm{d}\alpha\\&+\int_\sigma^t x^T(\alpha)y'(\alpha)\,\mathrm{d}\alpha+\sum_{n(\sigma)\leqslant i<n(t)}[x^T(\theta_i^+)y(\theta_i^+)-x^T(\theta_i)y(\theta_i)],\end{aligned}$$

where

$$n(t)=\min\{k\in\mathbb{N}:\theta_k\geqslant t\}.$$

If the differential equation satisfied by y is taken into account and the impulse conditions with respect to x and y in (1) and (5) are used, then

$$\begin{aligned}x^T(t)y(t)-x^T(\sigma)y(\sigma)={}&\int_\sigma^t\left[\int_{-\tau}^0 x^T(\alpha+s)\,\mathrm{d}_s\eta^T(\alpha,s)\right]y(\alpha)\,\mathrm{d}\alpha\\&-\int_\sigma^t x^T(\alpha)\mathrm{d}_\alpha\int_{-\tau}^0\eta^T(\alpha-s,s)y(\alpha-s)\,\mathrm{d}s\\&+\sum_{n(\sigma)\leqslant i<n(t)}\left[\sum_{-j\leqslant k<0}x^T(\theta_{i+k})A_{ik}^T\right]y(\theta_i^+)\\&-\sum_{n(\sigma)\leqslant i<n(t)}x^T(\theta_i)\sum_{-j\leqslant k<0}A_{(i-k)k}^T y(\theta_{i-k}^+)\\&+\int_\sigma^t f^T(\alpha)y(\alpha)\,\mathrm{d}\alpha+\sum_{n(\sigma)\leqslant i<n(t)}\beta_i^T y(\theta_i^+).\end{aligned}\tag{6}$$

Clearly, we may write (6) in an alternative form as

$$\begin{aligned}x^T(t)y(t)-x^T(\sigma)y(\sigma)={}&\int_\sigma^t\left[\int_{\alpha-\tau}^\alpha x^T(s)\,\mathrm{d}_s\eta^T(\alpha,s-\alpha)\right]y(\alpha)\,\mathrm{d}\alpha\\&-\int_\sigma^t x^T(s)\,\mathrm{d}_s\int_s^{s+\tau}\eta^T(\alpha,s-\alpha)y(\alpha)\,\mathrm{d}\alpha\\&+\sum_{n(\sigma)\leqslant i<n(t)}\left[\sum_{i\leqslant m<i-j}x^T(\theta_m)A_{i(m-i)}^T\right]y(\theta_i^+)\\&-\sum_{n(\sigma)\leqslant i<n(t)}x^T(\theta_i)\sum_{i\leqslant m<i+j}A_{m(i-m)}^T y(\theta_m^+)\\&+\int_\sigma^t f^T(\alpha)y(\alpha)\,\mathrm{d}\alpha+\sum_{n(\sigma)\leqslant i<n(t)}\beta_i^T y(\theta_i^+).\end{aligned}\tag{7}$$

The next step is to change the order of integration for which the following lemma, extracted from [8], is useful. Although the lemma is given for continuous functions, it is not difficult to see that it is also valid for piecewise left continuous functions.

Lemma 2. *If* $x \in PLC([c,d],\mathbb{R}^n)$, $y \in PLC([a,b],\mathbb{R}^n)$, *and* $\eta(t,s)$ *satisfies conditions* (*a*)–(*c*) *stated in Section* 2, *then*

$$\int_a^b \int_c^d x^T(s)\,\mathrm{d}_s\eta(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha = \int_c^d x^T(s)\,\mathrm{d}_s \int_a^b \eta(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha.$$

In view of Lemma 2 and the properties of $\eta(t,s)$ we may now interchange the order of integration in the first integral in (7) as in [8, p. 518] to see that

$$\begin{aligned}
&\int_\sigma^t \left[\int_{\alpha-\tau}^{\alpha} x^T(s)\,\mathrm{d}_s\eta^T(\alpha, s-\alpha)\right] y(\alpha)\,\mathrm{d}\alpha\\
&\quad = \int_{\sigma-\tau}^{\sigma} x^T(s)\,\mathrm{d}_s \int_\sigma^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha\\
&\qquad + \int_\sigma^{t-\tau} x^T(s)\,\mathrm{d}_s \int_s^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha\\
&\qquad + \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_s^{t} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha.
\end{aligned}$$

We also observe that

$$\begin{aligned}
&\sum_{n(\sigma)\leqslant i<n(t)} \left[\sum_{i\leqslant m<i-j} x^T(\theta_m)A^T_{i(m-i)}\right] y(\theta_i^+)\\
&\quad = \sum_{n(\sigma)-j\leqslant m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\leqslant i<m+j} A^T_{i(m-i)}y(\theta_i^+)\\
&\qquad + \sum_{n(\sigma)\leqslant m<n(t)-j} x^T(\theta_m) \sum_{m\leqslant i<m+j} A^T_{i(m-i)}y(\theta_i^+)\\
&\qquad + \sum_{n(t)-j\leqslant m<n(t)} x^T(\theta_m) \sum_{m\leqslant i<n(t)} A^T_{i(m-i)}y(\theta_i^+).
\end{aligned}$$

Using the above identities in (7) we arrive at

$$\begin{aligned}
x^T(t)y(t) - x^T(\sigma)y(\sigma) &= \int_{\sigma-\tau}^{\sigma} x^T(s)\,\mathrm{d}_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{\sigma}^{t-\tau} x^T(s)\,\mathrm{d}_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{s}^{t} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\quad - \int_{\sigma}^{t} x^T(s)\,\mathrm{d}_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\quad + \sum_{n(\sigma)-j \leqslant m < n(\sigma)} x^T(\theta_m) \sum_{n(\sigma) \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&\quad + \sum_{n(\sigma) \leqslant m < n(t)-j} x^T(\theta_m) \sum_{m \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&\quad + \sum_{n(t)-j \leqslant m < n(t)} x^T(\theta_m) \sum_{m \leqslant i < n(t)} A^T_{i(m-i)} y(\theta_i^+) \\
&\quad - \sum_{n(\sigma) \leqslant m < n(t)} x^T(\theta_m) \sum_{m \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&\quad + \int_{\sigma}^{t} f^T(\alpha)y(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma) \leqslant i < n(t)} \beta_i^T y(\theta_i^+).
\end{aligned} \tag{8}$$

Since

$$\begin{aligned}
&\int_{\sigma}^{t} x^T(s)\,\mathrm{d}_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\quad = \int_{\sigma}^{t-\tau} x^T(s)\,\mathrm{d}_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
&\qquad + \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{s}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n(\sigma) \leqslant m < n(t)} x^T(\theta_m) \sum_{m \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&\quad = \sum_{n(\sigma) \leqslant m < n(t)-j} x^T(\theta_m) \sum_{m \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\
&\qquad + \sum_{n(t)-j \leqslant m < n(t)} x^T(\theta_m) \sum_{m \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+),
\end{aligned}$$

we obtain from (8) that

$$\begin{aligned} x^T(t)y(t) - x^T(\sigma)y(\sigma) = & \int_{\sigma-\tau}^{\sigma} x^T(s)\,\mathrm{d}_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\ & - \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\ & + \sum_{n(\sigma)-j \leqslant m < n(\sigma)} x^T(\theta_m) \sum_{n(\sigma) \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\ & - \sum_{n(t)-j \leqslant m < n(t)} x^T(\theta_m) \sum_{n(t) \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+) \\ & + \int_{\sigma}^{t} f^T(\alpha)y(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma) \leqslant i < n(t)} \beta_i^T y(\theta_i^+). \end{aligned} \tag{9}$$

Define

$$\begin{aligned} \langle x(t), y(t)\rangle = & x^T(t)y(t) + \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)y(\alpha)\,\mathrm{d}\alpha \\ & + \sum_{n(t)-j \leqslant m < n(t)} x^T(\theta_m) \sum_{n(t) \leqslant i < m+j} A^T_{i(m-i)} y(\theta_i^+). \end{aligned} \tag{10}$$

Then we may write (9) in the form

$$\langle x(t), y(t)\rangle = \langle x(\sigma), y(\sigma)\rangle + \int_{\sigma}^{t} f^T(\alpha)y(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma) \leqslant i < n(t)} \beta_i^T y(\theta_i^+).$$

In particular if $x(t)$ is a solution of (3), i.e., when $f(t) \equiv 0$ and $\beta_i = 0$ for all $i \in \mathbb{N}$, then we obtain an important property

$$\langle x(t), y(t)\rangle = \langle x(\sigma), y(\sigma)\rangle = \text{constant}, \tag{11}$$

which generalizes a fundamental result to impulsive delay differential equations. Hence we may say that

Eq. (5) *is an adjoint of* (3) *with respect to function* $\langle x, y\rangle$ *given in* (10).

It can be shown that the adjoint of (3) is also (5), i.e., the equations are mutually adjoint of each other.

Remark. If there is no impulse effect then the second line in (10) disappears and the function coincides with the one used in [8].

Definition 3. A matrix solution $X(t, \alpha)$ of (3) satisfying $X(\alpha, \alpha) = I$ and $X(t, \alpha) = 0$ for $t < \alpha$ is called a fundamental matrix of Eq. (3). A matrix solution $Y(t, \alpha)$ of (5) satisfying $Y(\alpha, \alpha) = I$ and $Y(t, \alpha) = 0$ for $t > \alpha$ is said to be a fundamental matrix of Eq. (5).

The next theorem is of theoretical importance as it relates the solutions of (1) with the fundamental matrix $X(t, \alpha)$.

Theorem 4. *Let $X(t,\alpha)$ be a fundamental matrix of* (3) *and $\sigma \geqslant 0$ a real number. If $x(t)$ is a solution of* (1), *then*

$$\begin{aligned} x(t) = {} & X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} \mathrm{d}_s \left[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha, s-\alpha)\,\mathrm{d}\alpha \right] x(s) \\ & + \int_{\sigma}^{t} X(t,\alpha) f(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma)\leqslant i<n(t)} X(t,\theta_i^+)\beta_i \\ & + \sum_{n(\sigma)-j\leqslant m<n(\sigma)} \left[\sum_{n(\sigma)\leqslant i<m+j} X(t,\theta_i^+)A_{i(m-i)} \right] x(\theta_m). \end{aligned}$$

Proof. Replacing $y(\alpha)$ by the fundamental matrix $Y(\alpha,t)$ in (9), we get

$$\begin{aligned} x^T(t)Y(t,t) - x^T(\sigma)Y(\sigma,t) = {} & \int_{\sigma-\tau}^{\sigma} x^T(s)\,\mathrm{d}_s \int_{\sigma}^{s+\tau} \eta^T(\alpha, s-\alpha)Y(\alpha,t)\,\mathrm{d}\alpha \\ & - \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{t}^{s+\tau} \eta^T(\alpha, s-\alpha)Y(\alpha,t)\,\mathrm{d}\alpha \\ & + \sum_{n(\sigma)-j\leqslant m<n(\sigma)} x^T(\theta_m) \sum_{n(\sigma)\leqslant i<m+j} A_{i(m-i)}^T Y(\theta_i^+,t) \\ & - \sum_{n(t)-j\leqslant m<n(t)} x^T(\theta_m) \sum_{n(t)\leqslant i<m+j} A_{i(m-i)}^T Y(\theta_i^+,t) \\ & + \int_{\sigma}^{t} f^T(\alpha)Y(\alpha,t)\,\mathrm{d}\alpha + \sum_{n(\sigma)\leqslant i<n(t)} \beta_i^T Y(\theta_i^+,t). \end{aligned} \tag{12}$$

Since the terms in the second and the fourth line in (12) become zero, we easily get

$$\begin{aligned} x(t) = {} & Y^T(\sigma,t)x(\sigma) + \int_{\sigma-\tau}^{\sigma} \mathrm{d}_s \left[\int_{\sigma}^{s+\tau} Y^T(\alpha,t)\eta(\alpha, s-\alpha)\,\mathrm{d}\alpha \right] x(s) \\ & + \int_{\sigma}^{t} Y^T(\alpha,t) f(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma)\leqslant i<n(t)} Y^T(\theta_i^+,t)\beta_i \\ & + \sum_{n(\sigma)-j\leqslant m<n(\sigma)} \left[\sum_{n(\sigma)\leqslant i<m+j} Y^T(\theta_i^+,t)A_{i(m-i)} \right] x(\theta_m). \end{aligned} \tag{13}$$

On the other hand, replacing $x(t)$ by $X(t,\sigma)$ in (13) and using the properties $X(t,\sigma)=0$ for $t<\sigma$ and $X(\sigma,\sigma)=I$, we see that

$$X(t,\alpha) = Y^T(\alpha,t). \tag{14}$$

In view of (13) and (14) we have

$$
\begin{aligned}
x(t) = {} & X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} \mathrm{d}_s \left[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha, s-\alpha)\,\mathrm{d}\alpha \right] x(s) \\
& + \int_{\sigma}^{t} X(t,\alpha) f(\alpha)\,\mathrm{d}\alpha + \sum_{n(\sigma)\leqslant i<n(t)} X(t,\theta_i^+)\beta_i \\
& + \sum_{n(\sigma)-j\leqslant m<n(\sigma)} \left[\sum_{n(\sigma)\leqslant i<m+j} X(t,\theta_i^+)A_{i(m-i)} \right] x(\theta_m). \qquad \square \qquad (15)
\end{aligned}
$$

The representation of solutions of the adjoint equation (5) can be obtained in a similar manner.

Theorem 5. *Let* $Y(t,\alpha)$ *be a fundamental matrix of* (5) *and* $\sigma \geqslant 0$ *a real number. If* $y(t)$ *is a solution of* (5), *then*

$$
\begin{aligned}
y(t) = {} & Y(t,\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} Y(t,\beta)\,\mathrm{d}_\beta \int_{\sigma}^{\beta+\tau} \eta^T(\alpha,\beta-\alpha)y(\alpha)\,\mathrm{d}\alpha \\
& + \sum_{n(\sigma)-j\leqslant m<n(\sigma)} Y(t,\theta_m) \sum_{n(\sigma)\leqslant i<m+j} A_{i(m-i)}^T y(\theta_i^+).
\end{aligned}
$$

4. Auxiliary assertions

In this section, we prove that if the Perron condition is fulfilled then the fundamental matrix $X(t,\alpha)$ of equation (3) is bounded. To do this, we first provide a lemma similar to the one proved by Bellman and Cooke [4] for differential equations without impulse and delay, see also [8] for a related result on delay differential equations.

Lemma 6. *If Eq.* (3) *verifies Perron condition, then there exists a constant C such that*

$$
\int_0^t \|X(t,\alpha)\|\,\mathrm{d}\alpha + \sum_{0\leqslant m<n(t)} \|X(t,\theta_m^+)\| < C \quad for \quad t \geqslant 0.
$$

Proof. In view of Theorem 4, the solution $x(t)$ satisfying (2) with $\phi(t) \equiv 0$ can be written as

$$
x(t) = \int_0^t X(t,\alpha)f(\alpha)\,\mathrm{d}\alpha + \sum_{0\leqslant m<n(t)} X(t,\theta_m^+)\beta_m.
$$

From Perron condition, it follows that the function

$$
\int_0^t X(t,\alpha)f(\alpha)\,\mathrm{d}\alpha + \sum_{0\leqslant m<n(t)} X(t,\theta_m^+)\beta_m
$$

is bounded.

Let $\mathscr{X} = CB \times D$, where *CB* is the set of bounded functions $f \in C([0,\infty), \mathbb{R}^n)$ and *D* is the set of bounded sequences $\beta = \{\beta_m\}$, $\beta_m \in \mathbb{R}^n$, $m \in \mathbb{N}$. Clearly, $\mathscr{X}$ is a Banach space endowed with the norm $||(f,\beta)|| = \sup_{t\in[0,\infty)} \|f(t)\| + \sup_{m\in\mathbb{N}} \|\beta_m\|$. For fixed $t \in [0,\infty)$, consider the linear operator $U(f,\beta_m)$ on $\mathscr{X}$ defined by

$$U(f,\beta_m) = \int_0^t X(t,\alpha) f(\alpha)\,\mathrm{d}\alpha + \sum_{0\leqslant m<n(t)} X(t,\theta_m^+)\beta_m.$$

In a similar manner performed by Halanay [8] we may apply the Banach-Steinhaus theorem [11] to deduce that there exists a positive real number *C* such that

$$\int_0^t \|X(t,\alpha)\|\,\mathrm{d}\alpha + \sum_{0\leqslant m<n(t)} \|X(t,\theta_m^+)\| < C, \tag{16}$$

which completes the proof. □

Lemma 7. *If Eq.* (3) *verifies Perron condition*, *then there exists a constant M such that*

$$\|X(t,\alpha)\| < M, \quad t \geqslant \alpha \geqslant 0.$$

Proof. From (5) we see that

$$\frac{\partial Y^T(\alpha,t)}{\partial\alpha} = -\,\mathrm{d}_\alpha \int_{-\tau}^0 Y^T(\alpha-s,t)\eta(\alpha-s,s)\,\mathrm{d}s, \quad \alpha \neq \theta_i$$

$$\Delta Y^T(\theta_i,t) = -\left[Y^T(\theta_i,t)A_{i0} + \sum_{-j\leqslant k<0} Y^T(\theta_{i-k}^+,t)A_{(i-k)k}\right](I+A_{i0})^{-1}.$$

Integrating both sides from σ to *t*, we obtain

$$Y^T(\sigma,t) = I + \int_\sigma^t \mathrm{d}_\alpha \int_{-\tau}^0 Y^T(\alpha-s,t)\eta(\alpha-s,s)\,\mathrm{d}s - \sum_{n(\sigma)\leqslant i<n(t)} \Delta Y^T(\theta_i,t). \tag{17}$$

Observing that

$$Y^T(\theta_i,t) = Y^T(\theta_i^+,t)(I+A_{i0}) + \sum_{-j\leqslant k<0} Y^T(\theta_{i-k}^+,t)A_{(i-k)k},$$

it is not difficult to show that there is a positive real number $\mu_1 = \mu_1(\bar{\rho}_0,\rho_k)$ such that

$$\sum_{n(\sigma)\leqslant i<n(t)} \|\Delta Y^T(\theta_i,t)\| \leqslant \mu_1 \sum_{0\leqslant i<n(t)} \|Y^T(\theta_i^+,t)\|. \tag{18}$$

Further, by changing the order of integration in (17), we have

$$\int_\sigma^t \mathrm{d}_\alpha \int_{-\tau}^0 Y^T(\alpha-s,t)\,\eta(\alpha-s,s)\,\mathrm{d}s = \int_\sigma^{\sigma+\tau}\int_\sigma^r Y^T(r,t)\eta(r,\alpha-r)\,\mathrm{d}\alpha\,\mathrm{d}r + \int_{\sigma+\tau}^t\int_{r-\tau}^r Y^T(r,t)\eta(r,\alpha-r)\,\mathrm{d}\alpha\,\mathrm{d}r,$$

where $Y^T(r,t)=0$ for $r>t$ is used. It follows that

$$\begin{aligned}\left\|\int_\sigma^t \mathrm{d}_\alpha \int_{-\tau}^0 Y^T(\alpha-s,t)\eta(\alpha-s,s)\,\mathrm{d}s\right\| &\leqslant \int_\sigma^{\sigma+\tau}\int_\sigma^r \|Y^T(r,t)\eta(r,\alpha-r)\|\,\mathrm{d}\alpha\,\mathrm{d}r \\ &\quad + \int_{\sigma+\tau}^t\int_{r-\tau}^r \|Y^T(r,t)\eta(r,\alpha-r)\|\,\mathrm{d}\alpha\,\mathrm{d}r \\ &\leqslant \int_\sigma^t\int_{r-\tau}^r \|Y^T(r,t)\eta(r,\alpha-r)\|\,\mathrm{d}\alpha\,\mathrm{d}r,\end{aligned}$$

and so

$$\left\|\int_\sigma^t \mathrm{d}_\alpha \int_{-\tau}^0 Y^T(\alpha-s,t)\eta(\alpha-s,s)\,\mathrm{d}s\right\| \leqslant \gamma\int_0^t \|Y^T(r,t)\|\,\mathrm{d}r. \tag{19}$$

In view of (18) and (19) we see from (17) that

$$\|Y^T(\sigma,t)\|\leqslant 1+\gamma\int_0^t \|Y^T(r,t)\|\,\mathrm{d}r + \mu_1 \sum_{0\leqslant m<n(t)} \|Y^T(\theta_m^+,t)\|,$$

which, because of $Y^T(\sigma,t)=X(t,\sigma)$, gives the desired result. □

Finally, in the last section we are in a position to prove the main result of the paper.

5. Proof of Theorem 1

Let $x(t;\sigma,\phi)$ denote the solution of (3) satisfying $x(t+\sigma)=\phi(t)$ on $[-\tau,0]$, where $\sigma\geqslant 0$ is a real number and $\phi\in PLC([-\tau,0],\mathbb{R}^n)$. By $\|\phi\|_0$ we mean the supremum norm, that is,

$$\|\phi\|_0 = \sup_{t\in[-\tau,0]} \|\phi\|.$$

On the basis of Theorem 4, the solution $x(t;\sigma,\phi)$ is given by

$$x(t;\sigma,\phi) = X(t,\sigma)\phi(0) + \int_{-\tau}^{0} \mathrm{d}_s \left[\int_{\sigma}^{s+\sigma+\tau} X(t,\alpha)\eta(\alpha, s+\sigma-\alpha)\,\mathrm{d}\alpha\right]\phi(s)$$
$$+ \sum_{-j \leqslant m<0} \left[\sum_{n(\sigma)\leqslant i<m+n(\sigma)+j} X(t,\theta_i^+)A_{i(m+n(\sigma)-i)}\right]\phi(\theta_m).$$

Changing the order of integration and summation results in

$$x(t;\sigma,\phi) = X(t,\sigma)\phi(0) + \int_{\sigma}^{\sigma+\tau} X(t,\alpha) \int_{\alpha-\tau}^{\sigma} \mathrm{d}_s\eta(\alpha, s-\alpha)\phi(s)\,\mathrm{d}\alpha$$
$$+ \sum_{n(\sigma)\leqslant i<n(\sigma)+j} X(t,\theta_i^+) \sum_{i-j\leqslant m<n(\sigma)} A_{i(m-i)}\phi(\theta_m).$$

By virtue of Lemma 7, we obtain

$$\|x(t;\sigma,\phi)\| \leqslant M(1+\tau\gamma+j\rho)\|\phi\|_0,$$

where

$$\rho = \max\{\rho_k,\ k=-j,-j+1,\ldots,-1\}. \tag{20}$$

Thus, the zero solution of (3) is uniformly stable.

It remains to prove that the zero solution is uniformly attractive. That is,

$$\lim_{t\to\infty} x(t;\sigma,\phi) = 0, \tag{21}$$

uniformly with respect to σ and ϕ. To see this, let $\mu \geqslant \sigma$. Clearly,

$$x(t;\sigma,\phi) = X(t,\mu)x(\mu;\sigma,\phi) + \int_{\mu}^{\mu+\tau} X(t,\alpha) \int_{\alpha-\tau}^{\mu} \mathrm{d}_s\eta(\alpha, s-\alpha)x(s;\sigma,\phi)\,\mathrm{d}\alpha$$
$$+ \sum_{n(\mu)\leqslant i<n(\mu)+j} X(t,\theta_i^+) \sum_{i-j\leqslant m<n(\mu)} A_{i(m-i)}x(\theta_m;\sigma,\phi).$$

Integrating both sides with respect to μ over the interval $[\sigma, t]$, we get

$$(t-\sigma)x(t;\sigma,\phi) = \int_{\sigma}^{t} X(t,\mu)x(\mu;\sigma,\phi)\,\mathrm{d}\mu + \int_{\sigma}^{t}\left[\int_{\mu}^{\mu+\tau} X(t,\alpha)\xi(\mu,\alpha)\,\mathrm{d}\alpha\right]\mathrm{d}\mu$$
$$+ \int_{\sigma}^{t}\left[\sum_{n(\mu)\leqslant i<n(\mu)+j} X(t,\theta_i^+)\zeta(n(\mu),\theta_i)\right]\mathrm{d}\mu, \tag{22}$$

where

$$\xi(\mu, \alpha) = \int_{\alpha-\tau}^{\mu} \mathrm{d}_s \eta(\alpha, s-\alpha) x(s; \sigma, \phi),$$

and

$$\zeta(n(\mu), \theta_i) = \sum_{i-j \leqslant m < n(\mu)} A_{i(m-i)} x(\theta_m; \sigma, \phi).$$

On the other hand, one can easily verify that

$$\begin{aligned} &\int_{\sigma}^{t} \sum_{n(\mu) \leqslant i < n(\mu)+j} X(t, \theta_i^+) \zeta(n(\mu), \theta_i) \, \mathrm{d}\mu \\ &\quad = \sum_{n(\sigma) \leqslant i < n(\sigma)+j} \int_{\sigma}^{\theta_i} X(t, \theta_i^+) \zeta(n(\mu), \theta_i) \, \mathrm{d}\mu \\ &\qquad + \sum_{n(\sigma)+j \leqslant i < n(t)} \int_{\theta_{i-j}}^{\theta_i} X(t, \theta_i^+) \zeta(n(\mu), \theta_i) \, \mathrm{d}\mu. \end{aligned} \tag{23}$$

Using (23) and changing the order of integration in (22), we have

$$\begin{aligned} (t-\sigma) x(t; \sigma, \phi) = {} & \int_{\sigma}^{t} X(t, \mu) x(\mu; \sigma, \phi) \, \mathrm{d}\mu + \int_{\sigma}^{\sigma+\tau} \left[\int_{\sigma}^{\alpha} X(t, \alpha) \xi(\mu, \alpha) \, \mathrm{d}\mu \right] \mathrm{d}\alpha \\ & + \int_{\sigma+\tau}^{t} \left[\int_{\alpha-\tau}^{\alpha} X(t, \alpha) \xi(\mu, \alpha) \, \mathrm{d}\mu \right] \mathrm{d}\alpha \\ & + \sum_{n(\sigma) \leqslant i < n(\sigma)+j} \int_{\sigma}^{\theta_i} X(t, \theta_i^+) \zeta(n(\mu), \theta_i) \, \mathrm{d}\mu \\ & + \sum_{n(\sigma)+j \leqslant i < n(t)} \int_{\theta_{i-j}}^{\theta_i} X(t, \theta_i^+) \zeta(n(\mu), \theta_i) \, \mathrm{d}\mu, \end{aligned} \tag{24}$$

where $X(t, \alpha) = 0$ for $t < \alpha$ and $X(t, \theta_i^+) = 0$ for $n(t) < i$ have been used.

Denote $M_1 = M(1 + \tau\gamma + j\rho)$, where ρ is defined by (20). It is not difficult to see from (24) that

$$
\begin{aligned}
(t-\sigma)\|x(t;\sigma,\phi)\| \leqslant{}& M_1\|\phi\|_0 \int_\sigma^t \|X(t,\mu)\|\,\mathrm{d}\mu \\
&+ \int_\sigma^{\sigma+\tau} \left[\int_\sigma^\alpha \|X(t,\alpha)\|\|\xi(\mu,\alpha)\|\,\mathrm{d}\mu\right] \mathrm{d}\alpha \\
&+ \int_{\sigma+\tau}^t \left[\int_{\alpha-\tau}^\alpha \|X(t,\alpha)\|\|\xi(\mu,\alpha)\|\,\mathrm{d}\mu\right] \mathrm{d}\alpha \\
&+ \sum_{n(\sigma)\leqslant i<n(\sigma)+j} \int_\sigma^{\theta_i} \|X(t,\theta_i^+)\|\|\zeta(n(\mu),\theta_i)\|\,\mathrm{d}\mu \\
&+ \sum_{n(\sigma)+j\leqslant i<n(t)} \int_{\theta_{i-j}}^{\theta_i} \|X(t,\theta_i^+)\|\|\zeta(n(\mu),\theta_i)\|\,\mathrm{d}\mu. \qquad (25)
\end{aligned}
$$

Clearly, the stability of the trivial solution yields

$$\|\xi(\mu,\alpha)\| \leqslant \gamma \max_{s\in[\alpha-\tau,\mu]} \|x(s;\sigma,\phi)\| \leqslant \gamma M_1\|\phi\|_0.$$

Moreover, we have

$$\|\zeta(n(\mu),\theta_i)\| \leqslant \rho \max_{m\in[i-j,n(\mu)]} \|x(\theta_m;\sigma,\phi)\| \leqslant \rho M_1\|\phi\|_0.$$

Thus, inequality (25) leads to

$$
\begin{aligned}
(t-\sigma)\|x(t;\sigma,\phi)\| \leqslant{}& \gamma\tau^2 M M_1\|\phi\|_0 + \tau\gamma M_1\|\phi\|_0 \int_{\sigma+\tau}^t \|X(t,\alpha)\|\,\mathrm{d}\alpha \\
&+ \rho j v M M_1\|\phi\|_0 + v\rho M_1\|\phi\|_0 \sum_{n(\sigma)+j\leqslant i<n(t)} \|X(t,\theta_i^+)\| \\
&+ M_1\|\phi\|_0 \int_\sigma^t \|X(t,\mu)\|\,\mathrm{d}\mu,
\end{aligned}
$$

where v is as defined by (e). Making use of Lemma 6, we easily deduce that

$$(t-\sigma)\|x(t;\sigma,\phi)\| \leqslant [M M_1(\gamma\tau^2 + \rho j v) + M_1 C \max\{\tau\gamma, v\rho, 1\}]\|\phi\|_0,$$

and hence

$$\|x(t;\sigma,\phi)\| \leqslant \frac{M_2}{t-\sigma}\|\phi\|,$$

where $M_2 = M M_1(\gamma\tau^2 + \rho j v) + M_1 C \max\{\tau\gamma, v\rho, 1\}$. The last inequality obviously implies (21). □

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