Impulsive Control of the Population Dynamics

M.U. Akhmet[1], D. Aruğaslan[2], M. Beklioğlu[3]

[1]Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University (METU), 06531, Ankara, Turkey,
e–mail: marat@metu.edu.tr

[2]Department of Mathematics, METU, e–mail: aduygu@metu.edu.tr

[3]Department of Biology, METU, e–mail: meryem@metu.edu.tr

Abstract. We investigate the dynamics of the *Lotka-Volterra* system with variable time of impulses. Sufficient conditions are obtained for the existence of focus in the noncritical case. The focus-center problem in the critical case and the Hopf bifurcation are considered.

AMS Subject Classification. 34A37; 34C23; 34C25; 34D20

Key words. *Lotka-Volterra* system, focus, center, Hopf bifurcation, impulsive control.

1. Introduction

The *Lotka-Volterra* system describes the interaction of two species in an ecosystem, a prey and a predator. Since there are two species, this system involves two equations,

$$\begin{aligned} x' &= ax - bxy \\ y' &= -cy + dxy \end{aligned} \tag{1}$$

where x and y denote the prey and predator population densities, respectively, a (the growth rate of predator), b (the rate at which predators consume prey), c (the death rate of predator) and d (the rate at which predators increase by consuming prey) are positive constants. This system has only one positive equilibrium that is $(\frac{c}{d}, \frac{a}{b})$ as a center. However, having the equilibrium as center, this system is ecologically undesirable. In other words, the hypothesis of (1) does not seem to be in accordance with the observations [3].

The system (1) describes populations whose members can respond immediately to any change in the environment. But, in real populations, both prey and predator require reaction time lags. By introducing a time lag into system (1), instead of a center, the point of equilibrium may be either a stable focus or a stable node. Moreover, this point may be an unstable focus surrounded by a stable limit cycle [9].

The *Lotka-Volterra* population growth model does not assume human activities at all. We introduce human intervention by impulsive perturbation. In general, the appearence of the discontinuities can be explained by the fact that a development of a biological system may have sudden changes. It is natural that the obtained systems can be written in the form of impulsive differential equations [13], [14]. In this paper, our idea is to perturb system (1) by impulses at non-fixed moments of time. These impulses, in particular, may include man-made controls which are introduced when the state of species satisfies certain criterias. That is, we consider introducing or removing some members from the population as impulsive control. The approach of impulsive control was also proposed by Liu in [10, 11] and in the paper [12].

We mainly use the results which were obtained in [1] and [2]. One can verify that our sytems satisfy the properties of discontinuous dynamical systems described in [2], that is, the continuation of solutions on R, group property, continuous dependence of solutions on initial data and differentiability of solutions in initial data.

In Section 2, we formulate two problems; Problem (D) and Problem (U). In the next section, we investigate these problems. Lastly, the Hopf bifurcation for two systems which are associated with Problems D and U is considered in Section 4.

2. Formulation of the problems

In order to be more convenient, we first translate the equilibrium $(\frac{c}{d}, \frac{a}{b})$ to the origin by the linear transformation

$$\begin{bmatrix} x-\frac{c}{d} \\ y-\frac{a}{b} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2d\sqrt{ac}}{bc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This transformation takes system (1) into the form

$$\begin{aligned} x_1' &= -\sqrt{ac}\,x_2 - \tfrac{2d\sqrt{ac}}{c}x_1x_2 \\ x_2' &= \sqrt{ac}\,x_1 + 2dx_1x_2. \end{aligned} \tag{2}$$

We have new variables x_1 and x_2 possibly with negative values. But, the positiveness of the issue variables x and y in a neighborhood of the equilibrium $(\frac{c}{d}, \frac{a}{b})$ are certainly saved.

Clearly, systems (1) and (2) are qualitatively equivalent. Since $(\frac{c}{d}, \frac{a}{b})$ is a center of (1), the origin is a center of (2).

In what follows, we will consider how an impulsive perturbation may change the behaviour of the system (2) around the origin.

We introduce impulses into the system (2) with a more careful assumption that they are considered as impulsive control and we are sure that the more adequate explanation of the discontinuous population dynamics is a deal of future and is a deal of a closer collaboration of mathematicians and biologists. For that reason, we consider the impulsive control as the ability to instantly introduce or remove some members from the environment. It is acceptable and easily realizable as an ecological project. From this point of view, we formulate two problems to investigate:

Problem (D) Our objective is to bioregulate the *Lotka-Volterra* system by impulsive perturbation. Ecologically, it seems reasonable to control only the predator density. On the basis of this idea, we consider the impulsive action by means of removing some members of predators from the system. For example, if we have fish as predator (and Daphnia as prey) in a lake, the decrease in its density can be expressed by harvesting for commercial fishery. This type of dynamics can be modelled as follows;

$$\begin{aligned}
&x_1' = -\sqrt{ac}x_2 - \tfrac{2d\sqrt{ac}}{c}x_1x_2,\\
&x_2' = \sqrt{ac}x_1 + 2dx_1x_2, \quad (x_1,x_2) \notin \Gamma_1,\\
&\Delta x_1|_{(x_1,x_2)\in\Gamma_1} = 0,\\
&\Delta x_2|_{(x_1,x_2)\in\Gamma_1} = \kappa x_2,
\end{aligned} \tag{3}$$

where $\kappa < 0$ and Γ_1 is a half straight line in the second quadrant defined by the equation $x_2 = -\sqrt{3}x_1$ for $x_1 < 0$. When the solution meets the set Γ_1 at the time t_1, there exists a vertical jump, $\Delta x_2|_{t_1} = \kappa x_2(t_1) := x_2(t_1+) - x_2(t_1)$ going **down**.

We define determining the behaviour of solutions of system (3) around the origin as Problem (D). Furthermore, in Section 4, we will introduce a system with a small parameter μ associated with (3) and the Hopf bifurcation for that sytem is considered as Problem (DH).

Remark 2.1. Writing (3) in x, y coordinates, we obtain the following system

$$\begin{aligned}
&x' = ax - bxy,\\
&y' = -cy + dxy, \quad (x,y) \notin \tilde{\Gamma}_1,\\
&\Delta x|_{(x,y)\in\tilde{\Gamma}_1} = 0,\\
&\Delta y|_{(x,y)\in\tilde{\Gamma}_1} = \kappa(y - \tfrac{a}{b}),
\end{aligned}$$

where $\tilde{\Gamma}_1$ is a half-line defined by the equation $y - \frac{a}{b} = -\frac{\sqrt{3}d\sqrt{ac}}{bc}(x - \frac{c}{d})$ with $x < \frac{c}{d}$. So, we see that the corresponding impulsive control is only applied to the predator density.

Problem (U) Similar to the Problem (D), we can formulate Problem (U) for the system

$$\begin{aligned}
&x_1' = -\sqrt{ac}x_2 - \tfrac{2d\sqrt{ac}}{c}x_1x_2,\\
&x_2' = \sqrt{ac}x_1 + 2dx_1x_2, \quad (x_1,x_2) \notin \Gamma_2,\\
&\Delta x_1|_{(x_1,x_2)\in\Gamma_2} = 0,\\
&\Delta x_2|_{(x_1,x_2)\in\Gamma_2} = \kappa x_2,
\end{aligned} \tag{4}$$

where $\kappa < 0$ and Γ_2: $x_2 = -\sqrt{3}x_1$, $x_1 > 0$ is a straight line in the fourth quadrant. In this sytem, we control the predator density by introducing new members into the environment and thus we have a vertical jump going **up**. For the Hopf bifurcation, we shall define Problem (UH) in a manner similar to the Problem (DH).

3. Existence of foci and centers

3.1. Investigation of Problem (D)

Let $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. In system (3), we have discontinuity when $(x_1, x_2) \in \Gamma_1$. In polar coordinates r and ϕ, we have a jump when the angle is equal to $\frac{2\pi}{3} + 2\pi n$, $n \in Z$. Using polar transformation, we can write (3) in the following form:

$$\begin{aligned} &\tfrac{dr}{d\phi} = P(r,\phi), \qquad \phi \neq \tfrac{2\pi}{3} (\text{mod } 2\pi), \\ &\Delta r|_{\phi=\frac{2\pi}{3} (\text{mod } 2\pi)} = \lambda r, \\ &\Delta\phi|_{\phi=\frac{2\pi}{3} (\text{mod } 2\pi)} = \theta(\kappa), \end{aligned} \tag{5}$$

where $P(r,\phi) = \frac{(-\frac{2d}{c}\cos\phi + \frac{2d}{\sqrt{ac}}\sin\phi)\cos\phi\sin\phi\, r^2}{1+(\frac{2d}{\sqrt{ac}}\cos\phi+\frac{2d}{c}\sin\phi)\cos\phi\sin\phi\, r}$, $\lambda = \frac{1}{2}\sqrt{1+3(1+\kappa)^2} - 1$, $\theta(\kappa) = tan^{-1}(\frac{-\sqrt{3}\kappa}{4+3\kappa})$ and ϕ is ranged over the time-scale $\cup_{i=-\infty}^{\infty}(2\pi i + \frac{2\pi}{3} + \theta(\kappa), 2\pi(i+1) + \frac{2\pi}{3}]$. Clearly, the function P is 2π-periodic in ϕ and $P = o(r)$. Since (5) is a 2π-periodic system, we shall consider it only for $\phi \in [0, 2\pi] \setminus (\frac{2\pi}{3}, \frac{2\pi}{3} + \theta(\kappa)]$, i.e., the system

$$\begin{aligned} &\tfrac{dr}{d\phi} = P(r,\phi), \qquad \phi \neq \tfrac{2\pi}{3}, \\ &\Delta r|_{\phi=\frac{2\pi}{3}} = \lambda r, \\ &\Delta\phi|_{\phi=\frac{2\pi}{3}} = \theta(\kappa). \end{aligned} \tag{6}$$

System (6) is a time-scale differential equation. In order to obtain an impulsive differential equation, we use the $\psi-$ *substitution* method which is defined in [1]. The development of this method is given in [4]. Then one can obtain that the solution $r(\phi, r_0)$ of (6) starting at the point $(0, r_0)$ has the form

$$r(\phi, r_0) = \begin{cases} r_0 + \int_o^{\phi} P du, & \text{if } \ 0 \leq \phi \leq \frac{2\pi}{3}, \\ (1+\lambda)(r_0 + \int_0^{\frac{2\pi}{3}} P du) + \int_{\frac{2\pi}{3}+\theta(\kappa)}^{\phi} P du, & \text{if } \ \frac{2\pi}{3} + \theta(\kappa) < \phi \leq 2\pi. \end{cases}$$

Now, let us construct the Poincaré return map, $r(2\pi, r_0)$;

$$r(2\pi, r_0) = (1+\lambda) r_0 + (1+\lambda)\int_0^{\frac{2\pi}{3}} P du + \int_{\frac{2\pi}{3}+\theta(\kappa)}^{2\pi} P du. \tag{7}$$

From (7), we conclude that the origin of (3) is a stable focus if $1 + \lambda = \frac{1}{2}\sqrt{1+3(1+\kappa)^2} < 1$ and it is an unstable focus if $1 + \lambda > 1$. Then for the noncritical case, the following theorem is valid.

Theorem 3.1. If
(a) $-2 < \kappa < 0$, then the origin is a stable focus;
(b) $\kappa < -2$, then the origin is an unstable focus of system (3).

However, if $1 + \lambda = 1$, (i.e. if $\kappa = -2$) then we have the critical case and the origin is either a focus or a center. In what follows, we solve this problem of distinguishing between the focus and the center.

We can easily see that the angle $\theta(\kappa)$ is equal to $\frac{2\pi}{3}$ for $\kappa = -2$.

The solution $r(\phi, r_0)$ of (6), $r(0, r_0) = r_0$, for sufficiently small r, has the expansion [15]

$$r(\phi, r_0) = \textstyle\sum_{j=0}^{\infty} r_j(\phi) r_0^j, \tag{8}$$

with $\phi \in [0, 2\pi] \setminus (\frac{2\pi}{3}, \frac{4\pi}{3}]$, $r_o(\phi) = 0$, and $r_1(\phi) = 1$. Then, we have $r(2\pi, r_0) = \sum_{j=1}^{\infty} a_j r_0^j$ where $a_j = r_j(2\pi)$ and $a_1 = 1$. The function P also has the following expansion [15]

$$P(r, \phi) = \textstyle\sum_{j=2}^{\infty} P_j(\phi) r^j, \tag{9}$$

where

$$\begin{aligned} P_2(\phi) &= \left(-\tfrac{2d}{c}\cos\phi + \tfrac{2d}{\sqrt{ac}}\sin\phi\right)\cos\phi\sin\phi\,, \\ P_3(\phi) &= \left(\tfrac{\cos^2\phi - \sin^2\phi}{c\sqrt{ac}} + \tfrac{\cos\phi\sin\phi}{c^2} - \tfrac{\cos\phi\sin\phi}{ac}\right)4d^2\cos^2\phi\sin^2\phi. \end{aligned}$$

From the differential part of (6) and the expansion (9), one can find that

$$\begin{aligned} \tfrac{dr_2(\phi)}{d\phi} &= P_2(\phi) := \tilde{P}_2(\phi), \\ \tfrac{dr_3(\phi)}{d\phi} &= 2P_2(\phi) r_2(\phi) + P_3(\phi) := \tilde{P}_3(\phi) \end{aligned}$$

and similarly we define $\frac{dr_j(\phi)}{d\phi} := \tilde{P}_j(\phi)$ for $j = 4, 5, ...$

From the second equation of (6), we obtain that $r_j(\frac{4\pi}{3}) - r_j(\frac{2\pi}{3}) = 0$ for $j = 2, 3, ...$

Hence, the coefficients $r_j(\phi)$, $j = 2, 3, ...$ with $r_j(0) = 0$ are solutions of the system

$$\begin{aligned} &\tfrac{dr}{d\phi} = \tilde{P}_j(\phi), \qquad \phi \neq \tfrac{2\pi}{3}, \\ &\Delta r|_{\phi = \frac{2\pi}{3}} = 0, \\ &\Delta \phi|_{\phi = \frac{2\pi}{3}} = \tfrac{2\pi}{3}. \end{aligned} \tag{10}$$

As $a_j = r_j(2\pi)$, we can now evaluate a_j's in the expansion of $r(2\pi, r_0)$:

$$a_j = \textstyle\int_0^{\frac{2\pi}{3}} \tilde{P}_j(\phi) d\phi + \int_{\frac{4\pi}{3}}^{2\pi} \tilde{P}_j(\phi) d\phi \tag{11}$$

for $j = 2, 3, ...$

For the critical case, the sign of the first nonzero element of the sequence a_j determines what type of a singular point the origin is. The origin is a stable (unstable) focus if the first nonzero element is negative (positive). If all $a_j = 0$, $j = 2, 3, ...$ then the origin is a center [1]. That is why, we first need a_2 to solve this focus-center problem:

$$a_2 = \int_0^{\frac{2\pi}{3}} P_2(\phi)d\phi + \int_{\frac{4\pi}{3}}^{2\pi} P_2(\phi)d\phi = \frac{d\sqrt{3}}{2\sqrt{ac}}.$$

Since a_2 is positive, we have the following theorem.

Theorem 3.2. If $\kappa = -2$ then the origin of system (3) is an unstable focus.

3.2. Investigation of Problem (U)

Introducing polar coordinates, the system (4) can be written as follows;

$$\begin{aligned} &\tfrac{dr}{d\phi} = P(r,\phi), \qquad \phi \neq \tfrac{5\pi}{3}, \\ &\Delta r|_{\phi=\frac{5\pi}{3}} = \lambda r, \\ &\Delta \phi|_{\phi=\frac{5\pi}{3}} = \theta(\kappa), \end{aligned} \tag{12}$$

where P, λ and $\theta(\kappa)$ are the same as for system (6). For a solution $r(\phi, r_0)$, $r(0, r_0) = r_0$ of (12), the Poincaré return map is given by

$$r(2\pi, r_0) = (1+\lambda)r_0 + (1+\lambda)\textstyle\int_0^{\frac{5\pi}{3}} P du + \int_{\varphi(\kappa)}^{2\pi} P du \tag{13}$$

where $\frac{5\pi}{3} + \theta(\kappa) \equiv \varphi(\kappa) \pmod{2\pi}$.

Clearly, the noncritical case, that is $1+\lambda < 1$ or $1+\lambda > 1$, is treated similarly as in the investigation of Problem (D). But, the critical case, $1+\lambda = 1$, gives us a different result since the first element a_2 of the sequence a_j is negative;

$$a_2 = \int_0^{\frac{5\pi}{3}} P_2(\phi)d\phi + \int_{\frac{\pi}{3}}^{2\pi} P_2(\phi)d\phi = -\frac{d\sqrt{3}}{2\sqrt{ac}}.$$

Combining the results for noncritical and critical cases, we obtain the following assertion.

Theorem 3.3. If

(a) $-2 \leq \kappa < 0$, then the origin is a stable focus;

(b) $\kappa < -2$, then the origin is an unstable focus of (4).

4. Hopf bifurcation

Since the origin is a center, and not a focus, it is not possible to apply Hopf bifurcation theorem for system (2) which is the transformed *Lotka-Volterra* population growth model into $x_1\,, x_2$ coordinates [3]. But, with the impulsive control one can obtain the origin as a stable or an unstable focus, and hence, Hopf bifurcation can be investigated.

4.1. ***Problem*** (DH)

We introduce the following discontinuous dynamical system

$$\begin{aligned}
&x_1' = \mu x_1 - \sqrt{ac}x_2 - \tfrac{2d\sqrt{ac}}{c}x_1x_2,\\
&x_2' = \sqrt{ac}x_1 + \mu x_2 + 2dx_1x_2, \qquad (x_1, x_2) \notin \Gamma_1(\mu),\\
&\Delta x_1|_{(x_1,x_2)\in\Gamma_1(\mu)} = 0,\\
&\Delta x_2|_{(x_1,x_2)\in\Gamma_1(\mu)} = \kappa x_2,
\end{aligned} \tag{14}$$

where $\Gamma_1(\mu)$ is not a linear set and it is defined by the equation $x_2 = -\sqrt{3}x_1 + \mu x_1x_2$ for $x_1 < 0$. System (3) is associated with (14). In other words, (14) for $\mu = 0$ is the system (3) described in Section 2. In this system, μ appears to be an internal control parameter of the populations.

We shall also need the following system;

$$\begin{aligned}
&x_1' = \mu x_1 - \sqrt{ac}x_2,\\
&x_2' = \sqrt{ac}x_1 + \mu x_2, \qquad (x_1, x_2) \notin \Gamma_1,\\
&\Delta x_1|_{(x_1,x_2)\in\Gamma_1} = 0,\\
&\Delta x_2|_{(x_1,x_2)\in\Gamma_1} = \kappa x_2.
\end{aligned} \tag{15}$$

Using polar coordinates, (14) and (15) can be written as follows

$$\begin{aligned}
&\tfrac{dr}{d\phi} = \tfrac{\mu}{\sqrt{ac}}r + P(r, \phi, \mu),\ (r, \phi) \notin \Gamma_1(\mu),\\
&\Delta r|_{(r,\phi)\in\Gamma_1(\mu)} = \lambda r,\\
&\Delta \phi|_{(r,\phi)\in\Gamma_1(\mu)} = \theta(\kappa)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\tfrac{dr}{d\phi} = \tfrac{\mu}{\sqrt{ac}}r,\ \ \phi \neq \tfrac{2\pi}{3},\\
&\Delta r|_{\phi=\frac{2\pi}{3}} = \lambda r,\\
&\Delta \phi|_{\phi=\frac{2\pi}{3}} = \theta(\kappa),
\end{aligned} \tag{17}$$

respectively.

Now, the solution $r(\phi, r_0, \mu)$, $r(0, r_0, \mu) = r_0$ of (17) given by

$$r(\phi, r_0, \mu) = \begin{cases} exp(\frac{\mu}{\sqrt{ac}}\phi)r_0, & \text{if} \quad 0 \leq \phi \leq \frac{2\pi}{3},\\ (1+\lambda)exp(\frac{\mu}{\sqrt{ac}}(\phi - \theta(\kappa)))r_0, & \text{if} \quad \frac{2\pi}{3} + \theta(\kappa) < \phi \leq 2\pi. \end{cases}$$

implies that $r(2\pi, r_0, \mu) = (1+\lambda)exp(\frac{\mu}{\sqrt{ac}}(2\pi - \theta(\kappa)))r_0$.

Denote

$$q(\mu) = (1+\lambda)exp(\frac{\mu}{\sqrt{ac}}(2\pi - \theta(\kappa))).$$

Then we get $r(2\pi, r_0, \mu) = q(\mu)r_0$. $q(0) = 1$ and $q'(0) \neq 0$ are the necessary conditions [1] for the existence of periodical processes in system (16). It is easy to see that if $\lambda = 0$ (i.e. $\kappa = -2$) then $q(0) = 1$ and $q'(0) = \frac{4\pi}{3\sqrt{ac}} \neq 0$.

Applying the technique which is used in the paper [1], we can prove the folowing theorem:

Theorem 4.1. If $\kappa = -2$ then for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$ such that the solution $r(\phi, r_0, \delta(r_0))$ of (16) is periodic with period $T = \frac{4\pi}{3\sqrt{ac}} + o(|\mu|)$. Moreover, the closed trajectory is an unstable limit cycle.

4.2. Problem (UH)

We consider the system

$$(18)\qquad \begin{aligned} &x_1' = \mu x_1 - \sqrt{ac}x_2 - \tfrac{2d\sqrt{ac}}{c}x_1x_2,\\ &x_2' = \sqrt{ac}x_1 + \mu x_2 + 2dx_1x_2, \qquad (x_1,x_2) \notin \Gamma_2(\mu),\\ &\Delta x_1|_{(x_1,x_2)\in\Gamma_2(\mu)} = 0,\\ &\Delta x_2|_{(x_1,x_2)\in\Gamma_2(\mu)} = \kappa x_2, \end{aligned}$$

where $\Gamma_2(\mu)$ is a curve given by $x_2 = -\sqrt{3}x_1 + \mu x_1 x_2$ with $x_1 > 0$. Clearly, system (4) is associated with (18) . This system, in polar coordinates, is as follows:

$$(19)\qquad \begin{aligned} &\tfrac{dr}{d\phi} = \tfrac{\mu}{\sqrt{ac}}r + P(r,\phi,\mu),\ (r,\phi) \notin \Gamma_2(\mu),\\ &\Delta r|_{(r,\phi)\in\Gamma_2(\mu)} = \lambda r,\\ &\Delta \phi|_{(r,\phi)\in\Gamma_2(\mu)} = \theta(\kappa). \end{aligned}$$

Using the similar discussions made in Problem (DH) and using the paper [1], we can conclude the following result.

Theorem 4.2. If $\kappa = -2$ then for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$ such that the solution $r(\phi, r_0, \delta(r_0))$ of (19) is periodic with period $T = \frac{4\pi}{3\sqrt{ac}} + o(|\mu|)$. Moreover, the closed trajectory is a stable limit cycle.

5. Conclusion

Under the assumption that the coefficients a, b, c, d of the *Lotka-Volterra* system are positive, we may conclude that the complex behaviour of solutions entirely depends on the values of the coefficient κ which appears in the impulsive part of systems (3), (4), (14) and (18). That is, the problem of controllability of the *Lotka-Volterra* system by the proposed impulsive control is constructive.

References

[1]. M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, *Nonlinear Analysis,* **60**(2005), 163–178.

[2]. E. Akalın and M.U. Akhmet, The principles of B-smooth discontinuous flows, *Nonlinear Analysis,* **49**(2005), 981–995.

[3]. N. Minorsky, Nonlinear Oscillations, *Van Nostrand*, Princeton, 1962.

[4]. M.U. Akhmet and M. Turan, The differential equations on time scales through impulsive differential equations (submitted).

[5]. J.D. Murray, Mathematical Biology, *Springer*, New York, 2001.

[6]. J. Hale and H. Koçak, Dynamics and Bifurcations, *Springer-Verlag*, New York, 1991.

[7]. F.C. Hoppensteadt, Mathematical Methods of Population Biology, *Cambridge University.*

[8]. M. Scheffer, Ecology of Shallow Lakes, *Chapman and Hall*, 1998.

[9]. P.J. Wangersky and W.J. Cunningham, Time lag in prey-predator population models, *Ecology,* **vol.38**(Jan., 1957), 136–139.

[10]. X.Z. Liu and K. Rohlf, Impulsive control of Lotka-Volterra models, *IMA J.Math. Control Infor.,* **15**(1998), 269–284.

[11]. X.Z. Liu, Impulsive stabilization and control of chaotic systems, *Nonlinear Analysis,* **47**(2001), 1081–1092.

[12]. M.U. Akhmet, M. Beklioğlu, T. Ergenç and V.I. Tkachenko, On impulsive ratio-dependent predator-prey system with diffusion (submitted).

[13]. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, Theory of Impulsive Differential Equations, *World Scientific*, Singapore, 1989.

[14]. A.M. Samoilenko and N.A. Perestyuk, Impulsive Differential Equations, *World Scientific*, 1995.

[15]. M.U. Akhmet and N.A. Perestyuk, Asymptotic representation of solutions of regularly perturbed systems of differential equations with a nonclassical right-hand side, *Ukrainian Math.J.* **43**(1991), 1209–1214.