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The Principles of B-Smooth Discontinuous Flows

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Abstract—In this paper, we define B-smooth discontinuous dynamical systems which can be used as models of various processes in mechanics, electronics, biology, and medicine. We find sufficient conditions to guarantee the existence of such systems. These conditions are easy to verify. Appropriate examples are constructed. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

A book [1] edited by Anosov and Arnold considers two fundamentally different dynamical systems (DSs): flows and cascades. Roughly speaking, flows are DS with continuous time and cascades are DS with discrete time. One of the most important theoretical problem is to consider *discontinuous dynamical systems* (DDS). That is, systems whose trajectories are piecewise continuous curves. It is well-recognized (for example, see [2]) that the general notion of such systems was introduced by Pavlidis [3–5], although particular examples (the mathematical model of clock [6–8] and so on) had been discussed before. Some basic elements of the theory are given in [9–12]. Analysing the behavior of the trajectories, we can conclude that DDS combine features of vector fields and maps, they cannot be reduced to flows or cascades, but are close to flows since time is continuous. That is why we propose to call them also *discontinuous flows* (DF). Applications of DDS in mechanics, electronics, biology, and medicine were considered in [3–5,13–16]. Chaotic behavior of discontinuous processes was investigated in [14,17]. One must emphasize that DF are not *differential equations with discontinuous right side* which often have been accepted as DDS [18]. However, theoretical problems of nonsmooth dynamics and discontinuous maps [19–26] are also very close to the subject of our paper. One should also agree that *nonautonomous impulsive differential equations*, which were thoroughly described in [10] and [12], are not DF.

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Papers of Pavilidis and Rozhko [3–5,11] contain interesting practical and theoretical ideas concerning the DF. They formulated some important conditions on differential equations, but not all of them were used to prove basic properties of DF. Some aspects of DF on manifolds were considered in [27]. One must remark that the authors of the paper formulated conditions for the group property, but as is demonstrated by Example 8.3 of our paper those conditions do not guarantee it. In that paper, the *smooth impulsive flow* was claimed to be considered, but differentiable dependence, as well as continuous dependence, were not defined and investigated. Thus, one can say that the complexity of DF necessitates more careful investigation. Our article can be considered as an attempt to give a rigorous description of DF.

The paper embodies results that provide sufficient conditions for the existence of a *differentiable* DF. Since DF have specific smoothness of solutions we call these systems *B-differentiable DF*. Apparently, it is the first time when notions of B-continuous and B-differentiable dependence of solutions on initial values [28–32] are applied to describe DDS and sufficient conditions for the continuation of solutions and the group property are obtained. A very important novelty of the present paper is that we consider not only increasing time for discontinuous dynamical systems, but decreasing time, too. So, we introduce time symmetry in autonomous impulsive differential equations. A central auxiliary result of the paper, the construction of a new form of the general autonomous impulsive equation (system (1)), helps in achieving this aim. Effective methods of investigation of systems with variable time of impulsive actions were considered in [10,12,29–35].

Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ be the sets of all integers, natural, and real numbers, respectively. Denote by $\|\cdot\|$, the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider a set of strictly ordered real numbers $\{\theta_i\}$, where the set $\mathcal{A}$ of indices is an interval of $\mathbb{Z}/\{0\}$.

DEFINITION 1.1. *The set $\{\theta_i\}$ is said to be a sequence of β-type if the product $i\theta_i, i$, for all i and one of the following alternative cases holds,*

(a) $\{\theta_i\} = \emptyset$,
(b) $\{\theta_i\}$ *is a finite and nonempty set,*
(c) $\{\theta_i\}$ *is an infinite set, such that $|\theta_i| \to \infty$ as $|i| \to \infty$.*

From the definition, it follows immediately that a sequence of β-type does not have a finite accumulation point in $\mathbb{R}$.

DEFINITION 1.2. *A function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be from a space $\mathcal{PC}(\mathbb{R})$ if*

(1) *$\varphi(t)$ is left continuous on $\mathbb{R}$,*
(2) *there exists a sequence $\{\theta_i\}$ of β-type, such that φ is continuous if $t \neq \theta_i$ and φ has discontinuities of the first kind at the points θ_i.*

Particularly, $C(\mathbb{R}) \subset \mathcal{PC}(\mathbb{R})$.

DEFINITION 1.3. *A function $\varphi(t)$ is said to be from a space $\mathcal{PC}^1(\mathbb{R})$ if $\varphi' \in \mathcal{PC}(\mathbb{R})$.*

Let T be an interval in $\mathbb{R}$.

DEFINITION 1.4. *We denote by $\mathcal{PC}(T)$ and $\mathcal{PC}^1(T)$, the sets of restrictions of all functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}^1(\mathbb{R})$ on T, respectively.*

Let G be an open-connected subset of $\mathbb{R}^n$, G_r be an r-neighbourhood of G in $\mathbb{R}^n$ for a fixed $r > 0$, and $\hat{G} \subset G_r$ be an open-connected set. Denote as $\Phi : \hat{G} \longrightarrow \mathbb{R}$, a function from $C^1(\hat{G})$ and assume that a surface $\Gamma = \Phi^{-1}(0)$ is a subset of $\bar{G}$, where $\bar{G}$ denotes the closure of the set G in $\mathbb{R}^n$. Moreover, define a function $J : \Gamma_r \to \bar{G}$, where Γ_r is an r-neighbourhood of Γ. We shall need the following assumptions,

(C1) $\nabla\Phi(x) \neq 0$, $\forall\, x \in \Gamma$,
(C2) $J \in C^1(\Gamma_r), \det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma$.

One can see that the restriction $J|_\Gamma$ is a one-to-one function. Let also $\tilde{\Gamma} = J(\Gamma)$, $\tilde{\Gamma} \subset \bar{G}$. If $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$, $x \in \tilde{\Gamma}$, then $\tilde{\Gamma} = \{x \in G | \tilde{\Phi}(x) = 0\}$. It is easy to verify that $\nabla\tilde{\Phi}(x) \neq 0$,

$\forall x \in \tilde{\Gamma}$. Condition (C1) implies that for every $x_0 \in \Gamma$, there exists a number $j = \overline{1,n}$ and a function $\varphi_{x_0}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$, such that in a neigbourhood of x_0 the surface Γ is the graph of the function $x_j = \varphi_{x_0}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$. The same is true for every $x_0 \in \tilde{\Gamma}$.

REMARK 1.1. One can see from the description of Γ and $\tilde{\Gamma}$ that the surfaces are C^1 boundaryless $n-1$-dimensional manifolds [36].

Consider the following impulsive differential equation in the domain $D = [G \cup \Gamma \cup \tilde{\Gamma}] \backslash [(\bar{\Gamma}\backslash\Gamma) \cup (\bar{\tilde{\Gamma}}\backslash\tilde{\Gamma})]$.

$$\begin{gathered} x'(t) = f(x(t)), \qquad \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\}, \\ x(t+)|_{x(t-)\in\Gamma\wedge t\geq 0} = J(x(t-)), \\ x(t-)|_{x(t+)\in\tilde{\Gamma}\wedge t\leq 0} = J^{-1}(x(t+)). \end{gathered} \tag{1}$$

(C3) $f \in C^1(G_r)$.
(C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$, $\Gamma \cap (\bar{\tilde{\Gamma}}\backslash\tilde{\Gamma}) = \emptyset$, $(\bar{\Gamma}\backslash\Gamma) \cap \tilde{\Gamma} = \emptyset$.
(C5) $\langle \nabla\Phi(x), f(x)\rangle \neq 0$ if $x \in \Gamma$.
(C6) $\langle \nabla\tilde{\Phi}(x), f(x)\rangle \neq 0$ if $x \in \tilde{\Gamma}$.

2. EXISTENCE AND UNIQUENESS

DEFINITION 2.1. *A function $x(t) \in \mathcal{PC}^1(T)$ with a set of discontinuity points $\{\theta_i\} \subset T$ is said to be a solution of (1) on the interval $T \subset \mathbb{R}$ if it satisfies the following conditions,*

(i) *equation (1) is satisfied at each point $t \in T\backslash\{\theta_i\}$ and $x'(\theta_i-) = f(x((\theta_i)))$, $\theta_i \in T$, where $x'(\theta_i-)$ is the left-sided derivative;*
(ii) *$x(\theta_i+) = J(x((\theta_i))$ for all θ_i.*

THEOREM 2.1. *Assume that Conditions (C1)–(C6) hold. Then, for every $x_0 \in D$ there exists an interval $(a,b) \subset \mathbb{R}$, $a < 0 < b$, such that the solution $x(t) = x(t,0,x_0)$ of (1) exists on the interval.*

PROOF. To prove the theorem, we consider the following several cases.

(a) Assume that $x_0 \notin \Gamma\cup\tilde{\Gamma}$. Then, there exists a number $\epsilon > 0$, such that $B(x_0,\epsilon)\cap(\Gamma\cup\tilde{\Gamma}) = \emptyset$. Therefore, by the existence and uniqueness theorem [37], the solution exists and is unique on an interval (a,b) as a solution of the system,

$$y' = f(y). \tag{2}$$

(b) If $x_0 \in \Gamma$, then $x(0+) \in \tilde{\Gamma}$. There exists a number $\epsilon > 0$, such that $B(x(0^+),\epsilon) \cap \Gamma \neq 0$ and $x(t)$ can be continued to the right continuously. Let us consider decreasing t now. By Condition (C4), there exists a number $\epsilon > 0$, such that $B(x(0),\epsilon) \cap \tilde{\Gamma} \neq 0$ and $x(t)$ can be continued to the left continuously.
(c) We can discuss the case $x_0 \in \tilde{\Gamma}$ similarly to the previous one.

The uniqueness of the solution for all Cases (a)–(c) follows from the theorem on uniqueness for ordinary differential equations [37] and invertibility of the function J.

DEFINITION 2.2. *A solution $x(t) : [a,\infty) \to \mathbb{R}^n$, $a \in \mathbb{R}$, of (1) is said to be continuable to ∞.*

DEFINITION 2.3. *A solution $x(t) : (-\infty,b] \to \mathbb{R}^n$, $b \in \mathbb{R}$, of (1) is said to be continuable to $-\infty$.*

DEFINITION 2.4. *A solution $x(t)$ of (1) is said to be continuable on $\mathbb{R}$ if it is continuable to ∞ and to $-\infty$.*

DEFINITION 2.5. *A solution $x(t) = x(t,0,x_0)$ of (1) is said to be continuable to a set $S \subset \mathbb{R}^n$ as time decreases (increases) if there exists a moment $\xi \in \mathbb{R}$, such that $\xi \leq 0$ ($\xi \geq 0$) and $x(\xi) \in S$.*

Denote by $B(x_0,\xi) = \{x \in \mathbb{R}^n | \|x - x_0\| < \xi\}$, a ball with centre $x_0 \in \mathbb{R}^n$ and radius $\xi \in \mathbb{R}$.

The following theorem provides sufficient conditions for the continuation of solutions of (1).

THEOREM 2.2. *Assume the following.*

(a) *Every solution $y(t,0,x_0), x_0 \in D$, of (2) is either continuable to ∞ or continuable to Γ as time increases.*
(b) *For every $x \in \tilde{\Gamma}$ there exists a number ϵ_x, such that $\bar{B}(x,\epsilon_x) \cap \Gamma = \emptyset$.*
(c) $\inf_{(x,\epsilon_x)\in\Gamma\times(0,\infty)}(\epsilon_x/\sup_{B(x,\epsilon_x)} \|f(x)\|) = \theta > 0$.

Then, every solution $x(t) = x(t,0,x_0)$, $x_0 \in D$, of (1) is continuable to ∞.

PROOF. Let $x(\theta_i+) \in \tilde{\Gamma}$ for fixed i. Assume that there exists a number $\xi > \theta_i$, such that $\|x(\xi) - x(\theta_i+)\| = \epsilon_{x(\theta_i+)}$ (otherwise $x(t)$ is continuable to ∞). Then,

$$x(\xi) = x(\theta_i+) + \int_{\theta_i}^{\xi} f(x(s))\, ds$$

and $\epsilon_{x(\theta_i+)} \le M_{x(\theta_i+)}(\xi - \theta_i) \le M_{x(\theta_i+)}(\theta_{i+1} - \theta_i)$. The last inequality implies that $\theta_{i+1} - \theta_i \ge \theta$ for all i. That is, θ_i is a sequence of β-type if $\theta_i \ge 0$.

In a similar manner, one can prove that the following theorem is valid.

THEOREM 2.3. *Assume the following.*

(a) *Every solution $y(t,0,x_0)$, $x_0 \in D$, of (2) is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases.*
(b) *For every $x \in \Gamma$, there exists a number $\epsilon_x > 0$, such that $\bar{B}(x,\epsilon_x) \cap \tilde{\Gamma} = \emptyset$.*
(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times(0,\infty)}(\epsilon_x/\sup_{B(x,\epsilon_x)} \|f(x)\|) = \theta > 0$.

Then, every solution $x(t) = x(t,0,x_0)$, $x_0 \in D$, of (1) is continuable to $-\infty$.

Theorems 2.2 and 2.3 imply that the following assertion is valid.

THEOREM 2.4. *Assume the following.*

(a) *Every solution $y(t,0,x_0)$, $x_0 \in D$, of (2) satisfies the following conditions.*
  (a1) *It is continuable either to ∞ or to Γ as time increases.*
  (a2) *It is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases.*
(b) *For every $x \in \tilde{\Gamma}$, there exists a number $\epsilon_x > 0$, such that $\bar{B}(x,\epsilon_x) \cap \Gamma = \emptyset$.*
(b′) *For every $x \in \Gamma$, there exists a number $\tilde{\epsilon}_x > 0$, such that $\bar{B}(x,\tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$.*
(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times(0,\infty)(\epsilon_x/\sup_{\bar{B}(x,\epsilon_x)} \|f(x)\|)} > 0$.
(c′) $\inf_{(x,\tilde{\epsilon}_x)\in\Gamma\times(0,\infty)(\tilde{\epsilon}_x \sup_{B(x,\tilde{\epsilon}_x)} \|f(x)\|)} > 0$.

Then, every solution $x(t) = x(t,0,x_0)$, $x_0 \in D$, of (1) is continuable on $\mathbb{R}$.

Let us introduce a distance between two sets $A, B \subset \mathbb{R}^n$ as $\operatorname{dist}(A,B) = \inf\{\|a - b\| | a \in A, b \in B\}$.

Other sufficient conditions for the continuation of solutions of (1) are provided by the following theorems.

THEOREM 2.5. *Assume the following.*

(a) *Every solution $y(t,0,x_0)$, $x_0 \in D$, of (2) satisfies the following conditions.*
  (a1) *It is continuable either to ∞ or to Γ as t increases.*
  (a2) *It is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as t decreases.*
(b) $\sup_D |f(x)| < +\infty$.
(c) $\operatorname{dist}(\Gamma, \tilde{\Gamma}) > 0$.

Then, a solution $x(t,0,x_0), x_0 \in D$, of (1) is continuable on $\mathbb{R}$.

PROOF. Fix $x_0 \in D$ and let $x(t) = x(t,0,x_0)$ be the solution of (1). According to Definition 1.1, we shall consider the following three cases.

(A) If x(t) is a continuous solution of (1), then it is a solution of (2), and hence, is continuable on $\mathbb{R}$.

(B) Denote by $\theta_{\max}$ and $\theta_{\min}$ the maximal an minimal elements of the set $\{\theta_i\}$, respectively. Consider $t \geq \theta_{\max}$. By the condition on J, the value $x(\theta_{\max}+) = J(x(\theta_{\max}-)) \in D$ and the solution $x(t) = y(t, \theta_{\max}, x(\theta_{\max}+))$, where y is the solution of (2) and is continuable to ∞. For $t \leq \theta_{\min}$ one can apply the same arguments to show that x(t) is continuable to $-\infty$.

(C) Three alternatives exists. Let us consider them in turn.

(c_1) If the sequence $\{\theta_i\}$ has a maximal element $\theta_{\max} \in R$, then using (B), it is easy to prove that $x(t)$ is continuable to ∞. Let t be decreasing. We have that

$$x(\theta_i+) = x(\theta_{i+1}) + \int_{\theta_{i+1}}^{\theta_i} f(x(s))\, ds. \tag{3}$$

Denote $\sup_D |f(x)| = M$ and $\operatorname{dist}(\Gamma, \tilde{\Gamma}) = \alpha$. Then, (3) impies that $\alpha/M \leq (\theta_{i+1} - \theta_i)$. Hence, $\alpha/M(i - i_0) \geq (\theta_i - \theta_{i_0})$, where i_0 is fixed. The last inequality shows that $\theta_i \to -\infty$ as $i \to -\infty$. Thus, $x(t)$ is continuable to $-\infty$.

(c_2) Assume that the sequence $\{\theta_i\}$ has a minimal element $\theta_{\min} \in R$. Then, the arguments of (B) indicate that $x(t)$ is continuable to $-\infty$. For increasing t, we have that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))\, ds, \tag{4}$$

$\alpha/M \leq (\theta_{i+1} - \theta_i)$ or $\alpha/M(i - i_0) \leq (\theta_i - \theta_{i_0})$, where i_0 is fixed. Hence, $\theta_i \to \infty$ as $i \to \infty$. That is, $x(t)$ is continuable to ∞.

(c_3) Assume that $\{\theta_i\}$ has neither a minimal nor a maximal element. The result for this case follows from (c_1) and (c_2). The proof is complete.

THEOREM 2.6. *Assume the following.*

(a) *Every solution $y(t, 0, x_0), x_0 \in D$, of (2) is continuable either to ∞ or to Γ as time increases.*

(b) *There exists a neighbourhood S of Γ in D, such that*

(b1) $\operatorname{dist}(\Gamma, \partial S) > 0$;

(b2) $\sup_S \|f(x)\| < \infty$;

(b3) $\tilde{\Gamma} \cap S = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0)$, $x_0 \in D$, of (1) is continuable to ∞.

PROOF. Denote $d = \operatorname{dist}(\Gamma, \partial S)$ and $M = \sup_S \|f(x)\|$. For fixed i one can see that

$$x(\theta_{i+1}) = x(\theta_i+) + \int_{\theta_i}^{\theta_{i+1}} f(x(s))\, ds.$$

Condition (b3) implies that $d < |x(\theta_{i+1}) - x(\theta_i+)| \leq M(\theta_{i+1} - \theta_i)$. Thus, $\theta_{i+1} - \theta_i \geq d/M > 0$ for all i. Further discussion is fully analogous to the proof of the last Theorem.

Similarly, one can prove that the following assertion is valid.

THEOREM 2.7. *Assume the following.*

(a) *Every solution $y(t, 0, x_0), x_0 \in D$, of (2) is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases.*

(b) *There exists a neighbourhood $\tilde{S}$ of $\tilde{\Gamma}$ in D, such that*

(b1) $\operatorname{dist}(\tilde{\Gamma}, \partial\tilde{S}) > 0$;

(b2) $\sup_{\tilde{S}} \|f(x)\| < \infty$;

(b3) $\Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t, 0, x_0)$ $x_0 \in D$, of (1) is continuable to $-\infty$.

Using the conditions of both Theorems 2.6 and 2.7 one can formulate the following assertion.

THEOREM 2.8. *Assume the following.*

(a) *Every solution $y(t,0,x_0), x_0 \in D$, of (2) satisfies the following conditions.*
   (a1) *It is continuable either to ∞ or to Γ as time increases.*
   (a2) *It is continuable either to $-\infty$ or to $\tilde{\Gamma}$ as time decreases.*

(b) *There exists a neighbourhoods S and $\tilde{S}$ of Γ and $\tilde{\Gamma}$ in D, respectively, such that*
   (b1) $\operatorname{dist}(\Gamma, \partial S) > 0$, $\operatorname{dist}(\tilde{\Gamma}, \partial \tilde{S}) > 0$;
   (b2) $\sup_{S \cup \tilde{S}} \|f(x)\| < \infty$;
   (b3) $\tilde{\Gamma} \cap S = \emptyset$, $\Gamma \cap \tilde{S} = \emptyset$.

Then, every solution $x(t) = x(t,0,x_0)$ $x_0 \in D$, of (1) is continuable on $\mathbb{R}$.

3. THE GROUP PROPERTY

Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (1). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\theta \in \mathbb{R}$ and introduce a function $\psi(t) = x(t+\theta)$.

LEMMA 3.1. *The set $\{\theta_i - \theta\}$ is a set of all solutions of the equation,*

$$\Phi\left(\psi\left(t\right)\right) = 0. \tag{5}$$

PROOF. We have that $\Phi(\psi((\theta_i - \theta)) = \Phi(x((\theta_i - \theta) + \theta)) = \Phi(x(\theta_i)) = 0$. Assume that $t = \varphi$ is a solution of (5), then $\Phi(x(\varphi + \theta)) = \Phi(\psi(\varphi)) = 0$. That is, $\varphi + \theta$ is one of the numbers $\{\theta_i\}$. Let $\varphi + \theta = \theta_j$, then $\varphi = \theta_j - \theta$. The lemma is proved.

The following condition is one of the main assumptions for DF.

(C7)
   (a) For every $x \in \Gamma$, there exists $\epsilon_x > 0$, such that a function $\operatorname{sign}\Phi(x)$ is constant in $[B(x,\epsilon_x) \cap G] \backslash \Gamma$.
   (b) For every $x \in \tilde{\Gamma}$, there exists $\epsilon_x > 0$, such that a function $\operatorname{sign}\Phi(x)$ is constant in $[B(x,\epsilon_x) \cap G] \backslash \tilde{\Gamma}$.

LEMMA 3.2. *Assume that (C1)–(C7) hold and $x(t) : (-\alpha, \alpha) \to \mathbb{R}^n$, $\alpha > 0$, is a solution of (2). Then, $x(0) \notin \Gamma$ and $x(0) \notin \tilde{\Gamma}$.*

PROOF. Assume, on the contrary, that $x(0) = x_0 \in \Gamma$. We have that

$$\begin{aligned}\Phi\left(x\left(t\right)\right) = \Phi\left(x\left(t\right)\right) - \Phi\left(x_0\right) &= \left\langle \nabla\Phi\left(x_0\right), x\left(t\right) - x_0\right\rangle + o\left(\left\|x\left(t\right) - x_0\right\|\right) \\ &= \left\langle \nabla\Phi\left(x_0\right), f\left(x_0\right)t + o\left(|t|\right)\right\rangle + o\left(\left\|f\left(x_0\right)\right\|t + o\left(|t|\right)\right) \\ &= \left\langle \nabla\Phi\left(x\left(0\right)\right), f\left(x\left(0\right)\right)\right\rangle t + o\left(|t|\right).\end{aligned}$$

By Condition (C7) function $\operatorname{sign}\Phi(x(t))$ has a constant value for sufficiently small $|t|$. This contradiction proves our lemma for Γ. For $\tilde{\Gamma}$, the proof is similar.

LEMMA 3.3. *Assume that (C1)–(C7) hold. Then, $x(-t,0,x(t,0,x_0)) = x_0$ for all $x_0 \in D, t \in \mathbb{R}$.*

PROOF. Consider only $t > 0$, as $t < 0$ is very similar to the first case and $t = 0$ is primitive. If the set $\{\theta_i\}$ is empty then proof follows immediately from the assertion for DS [1]. One can see that it remains to check the equality $x(\theta_i-, 0, x(\theta_i+)) = x(\theta_i)$ is valid for all i, and the condition $x(-\theta_1, 0, x(\theta_1, 0, x_0)) = x_0$ is fulfilled. The first one is obvious because of invertibility of J. Let us consider the second one. Denote $x(t) = x(t,0,x_0)$, $\tilde{x}(t) = x(t,0,x(\theta_1))$. Since $x(\theta_1) \in \Gamma$, then by (C4), the solution $\tilde{x}$ moves along the trajectory of (2) for decreasing t. It could not meet $\tilde{\Gamma}$ if $t > -\theta_1$. Indeed, assume on the contrary that there exists θ, $-\theta_1 < \theta < 0$, moment where $\tilde{x}$ intersects $\tilde{\Gamma}$. Then, $\tilde{x}(\theta+) = x(\theta + \theta_1)$. We have obtained a contradiction to Lemma 3.2 since $x(t)$ is the solution of (2) in a neighbourhood of $t = \theta + \theta_1$. The lemma is proved.

LEMMA 3.4. *If $x(t):\mathbb{R}\to\mathbb{R}^n$ is a solution of (1), then $x(t+\theta), \theta\in\mathbb{R}$, is also a solution of (1).*

PROOF.

(a) From the last lemma, it follows that $\psi = x(t+\theta)$ is continuous on the interval $(\theta_i-\theta, \theta_{i+1}-\theta]$, $i\in\mathbb{Z}$. Fix $i\in\mathbb{Z}$, and consider $t\in(\theta_i-\theta,\theta_{i+1}-\theta]$. We have that $t+\theta\in(\theta_i,\theta_{i+1}]$ and in the same manner as for DS one can verify that $\psi'(t)=f(\psi(t))$. That is, equation (1) is satisfied by $x(t+\theta)$ for all $t\neq\theta_i-\theta$, $i\in\mathbb{Z}$, if we mean the left-sided derivatives.

(b) For fixed i, we have that $\psi(\theta_i-\theta+) = x(\psi(\theta_i-\theta+)+\theta) = x(\theta_i+) = J(x(\theta_i)) = J(\psi(\theta_i-\theta))$. Thus, one can see that the impulsive equation in (1) is also satisfied by $x(t+\theta)$ and this completes the proof.

Lemmas 3.1–3.4 imply that the following theorem is valid.

THEOREM 3.1. *Assume that Conditions (C1)–(C7) are fulfilled. Then,*

$$x\left(t_2, x\left(t_1, x_0\right)\right) = x\left(t_2+t_1, x_0\right), \tag{6}$$

for all $t_1, t_2\in\mathbb{R}$.

4. CONTINUOUS DEPENDENCE OF SOLUTIONS ON INITIAL VALUES

Let us introduce the following functions $\tau=\tau(x)$ and $\Psi=\Psi(x)$ which will be needed throughout the rest of the paper. Fix $\kappa\in\mathbb{R}$. Denote by $x(t)=x(t,\kappa,x)$, a solution of (2), $\tau=\tau(x)$, the moment of the first meeting of $x(t)$ with the surface Γ as t increases or decreases, and $\tilde\tau=\tilde\tau(x)$, the moment of the first meeting of $x(t)$ with the surface $\tilde\Gamma$ as t increases or decreases.

LEMMA 4.1. $\tau(x),\tilde\tau(x)\in C^1$.

PROOF. Let us consider τ, as for $\tilde\tau$, the proof is similar. Differentiating $\Phi(x(\tau,\kappa,x))=0$, and using (C5) one can get that

$$\frac{\partial\Phi\left(x\left(\tau,\kappa,x\right)\right)}{\partial\tau} = \frac{\partial\Phi\left(x\left(\tau,\kappa,x\right)\right)}{\partial x}\frac{dx(t)}{dt}\bigg|_{t=\tau} = \frac{\partial\Phi\left(x\left(\tau,\kappa,x\right)\right)}{\partial x} f\left(x\left(\tau,\kappa,x\right)\right)\neq 0.$$

The proof of the lemma follows immediately from the implicit function theorem and conditions on (2).

COROLLARY 4.1. *$\tau(x),\tilde\tau(x)$ are continuous functions.*

Now, let $x_1 = x(t,\tau,x(\tau)) + J(x(\tau))$, $\tilde x_1 = x(t,\tilde\tau, x(\tilde\tau)) + J^{-1}(x(\tilde\tau))$ be also solutions of (2). Define functions $\Psi(x)=x_1(\kappa)$, $\tilde\Psi(x)=\tilde x_1(\kappa)$. Similar to Lemma 4.1, one can show that the following assertion is valid.

LEMMA 4.2. $\Psi(x),\tilde\Psi(x)\in C^1$

Assume that $x^0(t):[a,b]\to\mathbb{R}^n, a\le 0\le b$, is a continuous solution of (1), $x^0(t)=x(t,0,x_0)$. Let $T=\{x\in G | x=x^0(t)$, for some $t\in[a,b]\}$. We shall show that the following lemma is valid.

LEMMA 4.3. *There exists an $\epsilon>0$, such that $T_\epsilon\cap\Gamma=\emptyset$, $T_\epsilon\cap\tilde\Gamma=\emptyset$, if $T\cap\Gamma=\emptyset$, $T\cap\tilde\Gamma=\emptyset$.*

PROOF. Assume, on the contrary, that there exists a sequence $x_n\in\Gamma$, $n\in\mathbb{N}$, such that $x_n\to T$, as $n\to\infty$. Since T is compact, there exists a subsequence, which we assume to be the sequence x_n itself, and a point $x_0\in T$, such that $x_n\to x_0$, as $n\to\infty$. As function Φ is continuous, either $x_0\in\Gamma$ or $x_0\in\bar\Gamma\backslash\Gamma$. But, $x_0\notin\Gamma$ by the assumption, and $x_0\notin\bar\Gamma\backslash\Gamma$ by the definition of D. This contradiction proves our lemma.

Now, we assume that solution $x^0(t)$ of (1) has an empty or nonempty set of discontinuity points, and all these points are interior in $[a,b]$. Denote by $x(t)=x(t,0,\bar x)$ another solution of (1).

DEFINITION 4.1. *The solution $x(t) : [a, b] \to \mathbb{R}^n$ is said to be in an ϵ-neighbourhood of $x^0(t)$ if*

1. *every point of discontinuity of $x(t)$ lies in the ϵ-neighbourhood of a point of discontinuity of $x^0(t)$;*
2. *for each $t \in [a, b]$ which is outside of the ϵ-neighbourhood of points of discontinuity of $x^0(t)$, the inequality $|x^0(t) - x(t)| < \epsilon$ holds;*
3. *the set of discontinuity points of $x^0(t)$ is empty, then the inequality $|x^0(t) - x(t)| < \epsilon$ is valid for all $t \in [a, b]$, and $x(t)$ does not have discontinuity points in $[a, b]$.*

DEFINITION 4.2. *Hausdorff's topology, built on the basis of all ϵ-neighbourhoods, $0 < \epsilon < \infty$, of piecewise solutions, will be called $B_{[a,b]}$-topology.*

THEOREM 4.1. *Assume that Conditions (C1)–(C6) are satisfied. Then, a solution $x^0(t) : [a, b] \to \mathbb{R}$ of (1) continuously depends on the initial value in $B_{[a,b]}$-topology.*

Moreover, if all points of discontinuity $\theta_i, i = -k, \ldots, -1, 1, \ldots, m$, of $x^0(t)$ are interior points in $[a, b]$, then, for sufficiently small $\|x_0 - \bar{x}\|$, the solution $x(t) = x(t, 0, \bar{x}), x(t) : [a, b] \to \mathbb{R}^n$, meets the surface Γ exactly $m + k$ times.

PROOF. Consider first the case when $x^0(t)$ is and $\theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are the points of $x^0(t)$, such that $a \leq \theta_{-k} < \cdots < \theta_{-1} \leq 0 \leq \theta_1 < \cdots < \theta_m \leq b$. We consider only the section $[0, b]$. The closeness of x(t) and $x^0(t)$ on $[a, 0]$ can be considered similarly. There are two cases:

(a) $x_0 \in \Gamma$ and
(b) $x_0 \notin \Gamma$.

We start with the second one.

Assume that $x^0(b) \notin \Gamma$. In other words, $t = b$ is not the discontinuity point of $x^0(t)$. For a positive number $\alpha \in R$, we shall construct a set G^α in the following way. Let $F_\alpha = \{(t, x) | t \in [0, b], \|x - x^0(t)\| < \alpha\}$, $G_i(\alpha)$, $i = \overline{0, m+1}$, be α-neighborhoods of points $(0, x_0)$, $(\theta_i, x(\theta_i))$, $i = \overline{1, m}$, $(b, x^0(b))$ in $\mathbb{R} \times \mathbb{R}^n$, respectively, and $\bar{G}_i(\alpha)$, $i = \overline{1, m}$, be α-neighborhoods of points $(\theta_i, x^0(\theta_i+))$, respectively. Denote

$$G^\alpha = F_\alpha \cup \left(\bigcup_{i=0}^{m+1} G_i(\alpha) \right) \cup \left(\bigcup_{i=1}^{m} \bar{G}_i(\alpha) \right).$$

Take $\alpha = h$ sufficiently small so that $G^h \subset G_t \times G_x$, where G_t is an interval, such that $[0, b] \subset G_t$. Fix $\epsilon \in \mathbb{R}$, $0 < \epsilon < h$.

1. In view of the theorem on continuous dependence on parameters [37], there exists $\bar{\delta}_m \in \mathbb{R}$, $0 < \bar{\delta}_m < \epsilon$, such that $\bar{G_m}(\bar{\delta}_m) \cap \Gamma = \emptyset$ and every solution $x_m(t)$ of (2), which starts in $\bar{G_m}(\bar{\delta}_m)$, is continuable to $t = b$, does not intersect Γ, and
$$\left\| x_m(t) - x^0(t) \right\| < \epsilon,$$
for those t.
2. The continuity of J implies that there exists $\delta_m \in \mathbb{R}$, $0 < \delta_m < \epsilon$, such that $(\kappa, x) \in G_m(\alpha_m)$ implies $(\kappa, x + J(x)) \in \bar{G}_m(\bar{\alpha}_m) \cap D$.
3. Using Corollary 4.1, continuous dependence of solutions on initial value, one can find $\bar{\delta}_{m-1}, 0 < \bar{\delta}_{m-1} < \epsilon$, such that a solution $x_{m-1}(t)$ of (2), which starts in $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap D$, $\bar{G}_{m-1}(\bar{\alpha}_{m-1}) \cap \Gamma \neq \emptyset$, intersects Γ in $G_m(\alpha_m)$ (we continue the solution $x_{m-1}(t)$ only to the moment of the intersection) and $\|x_{m-1}(t) - x^0(t)\| < \epsilon$ for all t from the common domain of $x_{m-1}(t)$ and $x^0(t)$.

Continuing the process for $m-2, m-3, \ldots, 1$, one can obtain a sequence of families of solutions of (2) $x_i(t), i = \overline{1, m}$, and a number $\delta \in \mathbb{R}, 0 < \delta < \epsilon$, such that a solution $x(t) = x(t, 0, \bar{x})$, which starts in $G_0(\delta) \cap D$, coincides over the first interval of continuity, except possibly, the δ_1-neighborhood of θ_1, with one of the solutions $x_1(t)$. Then, on the interval $[\theta_1, \theta_2]$, it coincides with one of the solutions $x_2(t)$, except possibly the δ_1-neighborhood of θ_1 and the δ_2-neighborhood of θ_2, etc. Finally, one can see that the integral curve of $x(t)$ belongs to G^ϵ, it has exactly k meeting points with Γ, θ_i^1, $i = \overline{1,m}$, $|\theta_i^1 - \theta_i| < \epsilon$ for all i and is continuable to $t = b$.

If $x^0(b) \in \Gamma$, then it is easy to see that $x(t)$ has either a discontinuity point $\theta_m^1 \le \theta_m$ or only $m-1$ discontinuity points θ_i^1, $i = \overline{1, m-1}$ in $[0,b]$.

Assume that $x_0 \in \Gamma$. In this case, $t = 0$ is a jump moment for $x^0(t)$ and $x(0+) \notin \Gamma$, that is $0 = \theta_1$. We assume that $x^0(t)$ has points of discontinuity $\theta_i, i = \overline{1,m}$. Similarly to the previous case, one can find the $\bar\delta_1$-neighborhood $\bar G(\bar\delta_1)$ of the point $(\theta_1, x(\theta_1^+))$ which serves the same role as $\bar\delta_1$ in the first case.

That is, if $(\kappa, x) \in \bar G(\bar\delta_1) \cap D$, then the solution $x(t)$ belongs to the ϵ-neighborhood of $x^0(t)$ in $B_{[0,b]}$-topology. Now, using Condition (C5) and continuity of f and J, it is easy to find $\delta, 0 < \delta < \epsilon$, such that every solution $x(t)$ of (1) which starts in δ-neighborhood of $(0, x_0)$ in D intersects Γ in $G_1(\delta_1) \cap D$.

For the case $a \le t \le 0$, we only should remark that similarly to $0 \le t \le b$ for a given $\epsilon > 0$, one can find δ', such that $(0,\bar x) \in G_0(\delta')$ implies that $x(t,0,\bar x)$ is in the ϵ-neighborhood of $x^0(t)$ in $B[a,0]$-topology. Finally, if $\delta(\epsilon) = \min(\delta,\delta')$ and $(0,\bar x) \in G_0(\delta(\epsilon))$, then $x(t,0,\bar x)$ is in the ϵ-neighborhood of $x^0(t)$ in $B[a,b]$-topology.

Let us assume that $x^0(t) : [a,b] \to \mathbb{R}^n$, $a \le 0 \le b$, is continuous solution of (1), $x^0(t) = x(t,0,x_0)$. Lemma 4.3 and theorem on continuity in initial value [37] imples that the theorem is valid for this case too. The theorem is proved.

5. B-EQUIVALENCE

Consider the solution $x^0(t) : [a,b] \to R^n, a \le 0 \le b$, of (1) again. This time, we assume that the set $\{\theta_i\}$ of points of discontinuity is not empty and they are interior points of $[a,b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$.

The following system of impulsive differential equations is very important in sequel,

$$\begin{aligned} y'(t) &= f(y), && t \ne \theta_i, \\ y(\theta_i+) &= W_i(y(\theta_i)), && \text{for } i > 0, \\ y(\theta_i) &= \tilde W_i(y(\theta_i+)), && \text{for } i < 0, \end{aligned} \tag{7}$$

where the function f is the same as in (1) and the maps $W_i, \tilde W_i$ will be defined below. The following condition will be nedeed in the rest of our paper. Without loss of generality, assume that there exists $r_1 \in \mathbb{R}, 0 < r_1 < r$, such that the r_1-neighbourhoods $G_i(r_1)$ of $(\theta_i, x^0(\theta_i))$ do not intersect each other. In view of (C5), one can suppose that r_1 is sufficiently small so that every solution of (2) which starts in $G_i(r_1)$ intersects Γ in $G_i(r_1)$ exactly once as t increases or decreases.

Fix $i = 1,\dots,m$ and let $\xi(t) = x(t,\theta_i,x), (\theta_i,x) \in G_i(r_1)$, be a solution of (2) and $\tau_i = \tau_i(x), \tau_i \ge \theta_i$ or $\tau_i < \theta_i$, be a meeting time of $\xi(t)$ with Γ and $\psi(t) = x(t,\tau_i,\xi(\tau_i) + J(\xi(\tau_i))$ be another solution of (2). Denote $W_i(x) = \psi(\theta_i)$. One can see that

$$W_i(x) = \int_{\theta_i}^{\tau_i} f(\xi(s))\,ds + J\left(x + \int_{\theta_i}^{\tau_i} f(\xi(s))\,ds\right) + \int_{\tau_i}^{\theta_i} f(\psi(s))\,ds \tag{8}$$

is a map of an intersection of the plane $t = \theta_i$ with $G_i(r_1)$ into the plane $t = \theta_i$. Similarly, for $i = -k,\dots,-1$, if we denote by $\xi(t) = x(t,\theta_i,x)$ and $\psi(t) = x(t,\tilde\tau_i,\xi(\tilde\tau_i) + J^{-1}(\xi(\tilde\tau_i))$ corresponding solutions of (2), then

$$\tilde W_i(x) = \int_{\theta_i}^{\tilde\tau_i} f(\xi(s))\,ds + J^{-1}\left(x + \int_{\theta_i}^{\tilde\tau_i} f(\xi(s))\,ds\right) + \int_{\tilde\tau_i}^{\theta_i} f(\psi(s))\,ds. \tag{9}$$

The functions $W_i, \tilde{W}_i$ are the maps Ψ and $\tilde{\Psi}$, respectively, defined in the beginning of this section with $\kappa = \theta_i$. Hence, Lemma 4.2 implies that all $W_i, \tilde{W}_i$ are continuously differentiable maps. It is obvious, that for sufficiently small r_1, $W_i(x), \tilde{W}_i(x) \in G_r$. Further, $\widehat{(\alpha,\beta]}$, $\{\alpha,\beta\} \subset R$, stands for an oriented interval, that is

$$\widehat{(\alpha,\beta]} = \begin{cases} (\alpha,\beta], & \text{if } \alpha \le \beta, \\ (\beta,\alpha], & \text{otherwise.} \end{cases}$$

Let $x(t)$ be a solution of (1), $x(t) = x(t, a, x(a))$, and $x(t)$ be close to $x^0(t)$ in $B_{[a,b]}$-topology so that x(t) has exactly $m-k$ points $\tau_i, i = -k, \ldots, -1, 1, 2, \ldots, m$, of discontinuity in $[a, b]$. Denote by $G(h)$ an h-neighbourhood of the point $x^0(0)$.

DEFINITION 5.1. *Systems (1) and (7) are said to be B-equivalent in G^{r_1} if there exists $h \in R$, $0 < h$, such that*

1. *for every solution $x(t)$, such that $x(0) \in G(h)$, the integral curve of $x(t)$ belongs to G^{r_1} and there exists a solution $y(t) = y(t, 0, x(0))$ of (7) which satisfies*

$$x(t) = y(t), \qquad t \in [a,b] \Big\backslash \bigcap_{i=-k}^{m} \widehat{(\tau_i, \theta_i]}.$$

Particularly,

$$x(\theta_i) = \begin{cases} y(\theta_i), & \text{if } \theta_i \le \tau_i, \\ y(\theta_i^+), & \text{otherwise,} \end{cases}$$

$$y(\tau_i) = \begin{cases} x(\tau_i), & \text{if } \theta_i \ge \tau_i, \\ x(\tau_i^+), & \text{otherwise.} \end{cases}$$

1. *Conversely, if (7) has a solution $y(t) = y(t, 0, x(0))$, $x(0) \in G(h)$, then there exists a solution $x(t) = x(t, 0, x(0))$ of (1) which has an integral curve in G^{r_1}, and (11) holds.*

LEMMA 5.1. *$x_0(t)$ is a solution of (1) and (7) simultaneously.*

PROOF. The proof follows immediately from (8) and (9).

THEOREM 5.1. *Assume that Conditions (C1)–(C6) are fulfilled. Then, systems (1) and (7) are B-equivalent in G^{r_1} if r_1 is sufficiently small.*

PROOF. Assume that $r_1 > 0$ is sufficiently small, so that $W_i, i = -k, \ldots, -1, 1, \ldots, m$, are defined. Let us check only the first part of Definition 5.1 as for the second one the proof is analogous. Theorem 4.1 implies that there exists a small $h, 0 < h < r_1$, such that if $\|\bar{x} - x_0\| < h$ and $\bar{x} \in D$, then the solution $x(t) = x(t, 0, \bar{x})$ belongs to $G^{r_1} \cap G_t \times D$, where $r_1 > 0$ has been chosen for W_i above. Assume that h is sufficiently small so that $x(t)$ has exactly $m + k - 1$ moments of discontinuity $t = \tau_i$, $i = -k, \ldots, -1, 1, \ldots, m$. Without loss of generality, we suppose that $\theta_i > \tau_i$ for all i and $x(0)$ is not the point of discontinuity. It is obvious that we need only to prove the theorem for $[0, b]$, as for $[a, 0]$ the proof is similar. Consider the solution $y(t) = x(t, 0, x(0))$ of (7). By the theorem on existence and uniqueness [37], the equality

$$x(t) = y(t), \tag{12}$$

on $[0, \tau_1]$ is valid. Since $(\tau_1, x(\tau_1)) \in G^{r_1}$, we have that

$$y(\theta_1+) = \int_{\tau_1}^{\theta_1} f(y(s))\, ds + W_i(y(\theta_1)) \tag{13}$$

is defined and moreover,

$$x(\theta_1) = x(\tau_1) + J(x(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))\, ds. \tag{14}$$

Using (12)–(14), one can obtain that

$$y(\theta_1+) = x(\tau_1) + \int_{\tau_1}^{\theta_1} f(y(s))\, ds + \int_{\theta_1}^{\tau_1} f(y(s))\, ds + J(y(\tau_1)) + \int_{\tau_1}^{\theta_1} f(x(s))\, ds = x(\theta_1).$$

Now, defining $x(t)$ and $y(t)$ as solutions of (2) with a common initial value $x(\theta_1)$, one can see that $x(t) = y(t)$, $t \in (\theta_1, \tau_2]$. Continuing in the same manner for all $t \in [0, b]$ one can show that $y(t)$ is continuable to $t = b$ and (10) holds. Moreover, it is easily seen that for sufficiently small r_1, the integral curve of $y(t)$ belongs to G_r. The theorem is proved.

6. DIFFERENTIABILITY OF SOLUTIONS IN INITIAL VALUE

Let us define derivatives of functions, $\tau_i(x), W_i(x)$, $i = 1\ldots, m$, and $\tilde{\tau}_i(x)$, $\tilde{W}_i(x)$, $i = -k, \ldots, -1$, which were described in Section 5, at the points $(x^0(\theta_i))$ and $(x^0(\theta_i+))$, respectively. We start with derivatives of $\tau_i(x)$ and $\tilde{\tau}_i(x)$. One should emphasize that $\tau_i, \tilde{\tau}_i$ are maps $\tau, \tilde{\tau}$ defined in Section 5 with $\kappa = \theta_i$. The equalities $\Phi(x(\tau_i(x))) = 0$, $\tilde{\Phi}(x(\tilde{\tau}_i(x))) = 0$ imply that

$$\Phi_x\left(x^0(\theta_i)\right) f\left(x^0(\theta_i)\right) d\tau_i + \sum_{j=1}^{n} \Phi_x\left(x^0(\theta_i)\right) \frac{\partial x^0(\theta_i)}{\partial x_j} dx_j,$$

$$\tilde{\Phi}_x\left(x^0(\theta_i+)\right) f\left(x^0(\theta_i+)\right) d\tau_i + \sum_{j=1}^{n} \tilde{\Phi}_x\left(x^0(\theta_i+)\right) \frac{\partial x^0(\theta_i+)}{\partial x_j} dx_j.$$

Using the last expression, one can obtain that

$$\begin{aligned} \frac{\partial \tau_i\left(x^0(\theta_i)\right)}{\partial x_j} &= -\frac{\Phi_x\left(x^0(\theta_i)\right)\left(\partial x^0(\theta_i)/\partial x_j\right)}{\Phi_x\left(x^0(\theta_i)\right) f\left(x^0(\theta_i)\right)}, \\ \frac{\partial \tilde{\tau}_i\left(x^0(\theta_i+)\right)}{\partial x_j} &= -\frac{\partial \Phi_x\left(x^0(\theta_i+)\right)\left(\partial x^0(\theta_i+)/\partial x_j\right)}{\tilde{\Phi}_x\left(x^0(\theta_i+)\right) f\left(x^0(\theta_i+)\right)}. \end{aligned} \tag{15}$$

Similarly, for W_i, the following expression is valid,

$$\begin{aligned} \frac{\partial W_i(x^0(\theta_i))}{\partial x_j} &= e_j + f\frac{\partial \tau_i}{\partial x_j} + \frac{\partial J}{\partial x}\left(e_j + f\frac{\partial \tau_i}{\partial x_j}\right) - f^+\frac{\partial \tau_i}{\partial x_j}, \\ \frac{\partial \tilde{W}_i(x^0(\theta_i+))}{\partial x_j} &= e_j + f^+\frac{\partial \tilde{\tau}_i}{\partial x_j} + \frac{\partial J^{-1}}{\partial x}\left(e_j + f\frac{\partial \tilde{\tau}_i}{\partial x_j}\right) - f^+\frac{\partial \tilde{\tau}_i}{\partial x_j}, \end{aligned} \tag{16}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ and the unit is j^{th} coordinate. Consider the solution $x^0(t) : [a, b] \to R^n, a \le 0 \le b$, of (1) again. This time, we assume that all points of discontinuity $\{\theta_i\}$ are interior points of $[a, b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$. Assume that $x^0(t) : [a, b] \to \mathbb{R}^n$ is the solution of (1) and (7). Moreover, systems (1) and (7) are B-equivalent in G^r and there exists $\delta \in R$, $\delta > 0$, such that every solution which starts in $G_0(\delta)$ is continuable to $t = b$. Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior. Denote by $x_j(t), j = \overline{1, n}$, a solution of (1) such that $x_j(t_0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0)$, $\xi \in \mathbb{R}$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let θ_i^j be the moments of discontinuity of $x_j(t)$. By Theorem 4.1, for sufficiently small $|\xi|$ the solution $x_j(t)$ is defined on $[a, b]$.

DEFINITION 6.1. *The solution $x^0(t)$ is said to be B-differentiable in $x_j^0, j = \overline{1,n}$, if*

(A) *there exist such constants $\nu_{ij}, i = -k, \dots, -1, 1, \dots, m$, that*

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|); \tag{17}$$

(B) *for all $t \in [a,b] \setminus \bigcup_{i=-k}^{m} \widehat{(\theta_i, \theta_i^j]}$, the following equality is satisfied*

$$x_j(t) - x^0(t) = u_j(t)\xi + o(|\xi|); \tag{18}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i, i = -k, \dots, -1, 1, \dots, m$.

(C) *for the case when $x^0(t)$ is continuous on $[a,b]$, and $|x_j^0|$, $j = \overline{1,n}$, is sufficiently small, the solution $x_j(t)$ does not have discontinuity points in $[a,b]$, and (18) valid, where $u_j(t)$ is a continuous on $[a,b]$ function. The pair $\{u_j, \{\nu_{ij}\}_i\}$ is said to be a B-derivative of $x^0(t)$ in initial value x_0^j on $[a,b]$ if the set of discontinuity points of $x^0(t)$ in $[a,b]$ is not empty.*

LEMMA 6.1. *Assume that Conditions (C1)–(C6), then, the solution $x^0(t)$ of (7) has B-derivatives in the initial value on $[a,b]$. Moreover,*

(1) *$u_j, j = \overline{1,n}$, are solutions of the linear system,*

$$\begin{aligned} \frac{du}{dt} &= f_x(x^0(t))u, && t \neq \theta_i, \\ u(\theta_i+) &= W_{ix}(x^0(\theta_i))u(\theta_i), && \text{if } i > 0, \\ u(\theta_i) &= \tilde{W}_{ix}(x^0(\theta_i+))u(\theta_i+), && \text{if } i < 0, \end{aligned}$$

with the initial conditions $u(t_0) = e_j$, $j = \overline{1,n}$, respectively and constants $\nu_{ij} = 0$, for all i, j.

PROOF. Fix $p = \overline{1,n}$. We shall prove the lemma only for the derivative in x_0^p and for $t \geq 0$. Let $y_p(t) = y(t, t_0, x_0 + \xi e_p, \mu_0)$. By the theorem on differentiability with respect to parameters [37], we have that $y_p(t) - x^0(t) = u_p(t)\xi + \rho(\xi)$, $\rho(\xi) = o(|\xi|)$, for all $t \in [0, \theta_1]$. Particularly, $y_p(\theta_1) - x^0(\theta_1) = u_p(\theta_1)\xi + \rho(|\xi|)$. Then, $y_p(\theta_1+) - x^0(\theta_1+) = W_1(y_p(\theta_1)) - W_1(x^0(\theta_1)) = W_{1x}(x^0(\theta_1)))(u_p(\theta_1)\xi + \rho(\xi)] + \bar{\rho}_1(\xi)$. Since $\bar{\rho}_1 = o(|\xi|)$, we have that $y_p(\theta_1+) - x^0(\theta_1+) = u_p(\theta_1+)\xi + \tilde{\rho}_1(\xi)$, where $\tilde{\rho}_1 = o(|\xi|)$. Denote by $U(t), U(\theta_1) = I$, the fundamental matrix of solutions of the system $u'(t) = f_x(x^0(t))$. Using the theorem from [37] again, one can obtain that for all $t \in (\theta_1, \theta_2]$ the following relation is true $y_p(t) - x^0(t) = U(t)(y_p(\theta_1+) - x^0(\theta_1+)) + \rho(y_p(\theta_1+) - x^0(\theta_1+)) = U(t)u_p(\theta_m+)\xi + \rho_2(\xi) = u_p(t)\xi + \rho_2(\xi)$, where $\rho_2 = o(|\xi|)$. Continuing the process we can prove that (18) is valid. Formula (17) involving constants ν_i^j is trivial. The lemma is proved.

LEMMA 6.2. *Assume that Conditions (C1)–(C6) are satisfied and the solution $x^0(t) : [a,b] \to R^n, a \leq 0 \leq b$, of (1) such that all points of discontinuity $\{\theta_i\}$ are interior points of $[a,b]$. That is, $a < \theta_{-k}$ and $\theta_m < b$. Then, the solution $x^0(t)$ of (1) has B-derivatives in the initial value on $[a,b]$. Moreover, $u_j(t), j = \overline{1,n}$, are, respectively, solutions of equation (19) with the initial conditions $u(t_0) = e_j, j = \overline{1,n}$, and*

$$\begin{aligned} \nu_{ij} &= -\frac{\Phi_x u_j(\theta_i)}{\Phi_x f}, && j = \overline{1,n}, \quad i = \overline{1,m}, \\ \nu_{ij} &= -\frac{\tilde{\Phi}_x\left(x^0(\theta_i+)\right)u_j(\theta_i+)}{\tilde{\Phi}_x\left(x^0(\theta_i+)\right)f\left(x^0(\theta_i+)\right)}, && j = \overline{1,n}, \quad i = \overline{-k,-1}. \end{aligned}$$

The proof of this lemma follows immediately Theorem 5.1, Lemma 6.1, and formulas (15),(16).

Lemma 4.3, Lemma 6.2, and theorem on differentiability of solutions of ordinary differential equations [37] imply that the following theorem is valid.

THEOREM 6.1. *Assume that Conditions (C1)–(C6) hold. Then, every solution $x^0(t) : [a,b] \to R^n, a \le 0 \le b$, of (1) is B-differentiable in initial value x_0, if the points of discontinuity are interior in $[a,b]$.*

REMARK 6.1. Higher-order smoothness of solutions for impulsive autonomous differential equations is considered in [30].

7. CONCLUSION

Let $G \subset \mathbb{R}^n$ be an open set and $\Gamma, \tilde{\Gamma}$ be disjoint subsets of $\bar{G}$ described in Section 1. Denote $D = [G \cup \Gamma \cup \tilde{\Gamma}] \backslash [(\bar{\Gamma} \backslash \Gamma) \cup (\bar{\tilde{\Gamma}} \backslash \tilde{\Gamma})]$.

DEFINITION 7.1. *We say that a B-smooth DF is a map $\phi : \mathbb{R} \times D \to D$, which satisfies the following properties.*

(I) *The group property:*
   (i) *$\phi(0,x) : D \to D$ is the identity;*
   (ii) *$\phi(t, \phi(s,x)) = \phi(t+s,x)$, is valid for all $t,s \in \mathbb{R}$ and $x \in D$.*

(II) *If $x \in D$ is fixed, then $\phi(t,x) \in \mathcal{P}C^1(\mathbb{R})$, and $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ for every discontinuity point θ_i of $\phi(t,x)$.*

(III) *The function $\phi(t,x)$ is B-differentiable in x on $[a,b] \subset \mathbb{R}$ for every $\{a,b\} \subset \mathbb{R}$, assuming that all discontinuity points of $\phi(t,x)$ are interior points of $[a,b]$.*

One can see that the system (1) defines a B-smooth DF provided Conditions (C1)–(C7) and the conditions of one of the continuation theorems are fulfilled.

DEFINITION 7.2. *We say that a DF is a map $\phi : \mathbb{R} \times D \to D$, which satisfies the Property (I) of Definition 7.1 and the following conditions are valid.*

(IV) *If $x \in D$ is fixed, then $\phi(t,x) \in \mathcal{P}C(\mathbb{R})$, and $\phi(\theta_i, x) \in \Gamma$, $\phi(\theta_i+, x) \in \tilde{\Gamma}$ for every discontinuity point θ_i of $\phi(t,x)$.*

(V) *The function $\phi(t,x)$ is B-continuous in $x \in D$ on $[a,b] \subset \mathbb{R}$ for every $\{a,b\} \subset \mathbb{R}$.*

Comparing definitions of the B-differentiability and the B-continuity one can conclude that every B-smooth DF is a DF.

8. EXAMPLES

EXAMPLE 8.1. Consider the following impulsive differential equation

$$\begin{aligned}
x_1' &= \alpha x_1 - \beta x_2, \\
x_2' &= \beta x_1 + \alpha x_2, \qquad \text{if } (x(t) \notin \Gamma \wedge t \ge 0) \vee \left(x(t) \notin \tilde{\Gamma} \wedge t \le 0\right), \\
x_1(t+) &= \sqrt{3} x_1(t-) - x_2(t-), \\
x_2(t+) &= x_1(t-) + \sqrt{3} x_2(t-), \qquad \text{if } x(t) \in \Gamma \wedge t \ge 0, \\
x_1(t-) &= \frac{\sqrt{3}}{4} x_1(t+) + \frac{1}{4} x_2(t+), \\
x_2(t-) &= -\frac{1}{4} x_1(t+) + \frac{\sqrt{3}}{4} x_2(t+), \qquad \text{if } x(t) \in \tilde{\Gamma} \wedge t \le 0,
\end{aligned}$$

where $\Gamma = \{(x_1,x_2) \mid x_2 = 1/2x_1, x_1 > 0\}$, $\tilde{\Gamma} = \{(x_1,x_2) \mid x_2 = \sqrt{3}/2x_1, x_1 > 0\}$, $D = \mathbb{R}^2 \backslash [\{(x_1,x_2) \mid 1/2x_1 < x_2 < \sqrt{3}/2x_1, x_1 > 0\} \cup (0,0)]$ constants α, β are positive. One can see that $\Phi(x) = x_2 - 1/2x_1$, $f(x) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2)$, $J(x) = (\sqrt{3}x_1 - x_2, x_1 + \sqrt{3}x_2)$. One can verify that the functions and the sets satisfy (C1)–(C7). Let us check if the conditions of Theorem 2.4 hold. Fix $x \in \tilde{\Gamma}$. Then, $\mathrm{dist}(x,\Gamma) = 1/2||x||$ and $||f(x)|| = \sqrt{(\alpha x_1 - \beta x_2)^2 + (\beta x_1 + \alpha x_2)^2} = \sqrt{\alpha^2 + \beta^2}||x||$. Thus, $\sup ||f||_{B(x,\epsilon_x)} = \sqrt{\alpha^2+\beta^2}(||x|| + 1/2||x||) = 3/2\sqrt{\alpha^2+\beta^2}||x||$, and

$$\inf_{\tilde{\Gamma} \times (0,\infty)} \frac{\epsilon_x}{\sup_{B(x,\epsilon_x)} ||f||} = \frac{2}{3\sqrt{\alpha^2+\beta^2}} > 0.$$

Hence, all conditions of a DF for the system are fulfilled.

EXAMPLE 8.2. Consider the following model for simple neural nets from [3]. We have modified its form according to the proposed equation (1).

$$
\begin{aligned}
&x_1' = x_2, \qquad x_2' = -\beta^2 x_1, \\
&p' = -\gamma p + x_1 + B_0, \qquad \text{if } (x(t) \notin \Gamma \wedge t \geq 0) \vee \left(x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right), \\
&x_1(t+) = x_2(t-), \qquad x_2(t+) = x_2(t-), \qquad p(t+) = 0, \quad \text{if } x(t) \in \Gamma \wedge t \geq 0, \\
&x_1(t-) = x_1(t+), \qquad x_2(t-) = x_2(t+), \qquad p(t-) = r, \quad \text{if } x(t) \in \tilde{\Gamma} \wedge t \leq 0,
\end{aligned}
$$

where $\beta, B_0 \in \mathbb{R}$ are constants, $\Gamma = \{(x_1, x_2, p) | p = r\}$, $\tilde{\Gamma} = \{(x_1, x_2, p) | p = 0\}$, $D = \{(x_1, x_2, p) | 0 \leq p \leq r, x_1^2 + x_2^2/\beta^4 < 1\}$, $\Phi(x) = p - r$, $f(x) = (x_2, \beta^2 x_1, -\gamma p + x_1 + B_0) J(x) = (x_1, x_2, r), \beta, \gamma, r > 0$, are constants and $B_0 > 0$, $B_0 - \gamma r + 1 > 0$. In the system, the variable $p(t)$ is a scalar input of a neural trigger and x_1, x_2, are other variables. The value of r is the threshold. One can verify that the functions and the sets satisfy (C1)–(C7) and the conditions of Theorem 2.5. That is, the system defines a DF.

EXAMPLE 8.3. Let us consider the following discontinuous system,

$$
\begin{aligned}
&x_1' = \alpha x_1 - \beta x_2, \\
&x_2' = \beta x_1 + \alpha x_2, \qquad \text{if } (x(t) \notin \Gamma \wedge t \geq 0) \vee \left(x(t) \notin \tilde{\Gamma} \wedge t \leq 0\right), \\
&x_1(t+) = k x_1(t-), \quad x_2(t+) = k x_2(t-), \qquad \text{if } x(t) \in \Gamma \wedge t \geq 0, \\
&x_1(t-) = \frac{1}{k} x_1(t+), \\
&x_2(t-) = -\frac{1}{k} x_2(t+), \qquad \text{if } x(t) \in \tilde{\Gamma} \wedge t \leq 0,
\end{aligned}
\tag{20}
$$

where $\Gamma = \{(x_1, x_2) | x_1^2 + x_2^2 = r_1\}$, $\tilde{\Gamma} = \{(x_1, x_2) | x_1^2 + x_2^2 = k r_1\}$, α, β, k are constants, such that $\alpha, \beta < 0$, $1 < k$. Assume that $D = \mathbb{R}^2$.

One can see that all Conditions (C1)–(C6) are valid for the system, and so are conditions of Theorem 2.5. But (C7) is not fulfilled, and it is easy to see that a solution $x(t, 0, x_0)$ of (20), which starts outside of $\tilde{\Gamma}$, does not satisfy the condition $x(-t, 0, x(t, 0, x_0)) = x_0$ for all t. Thus, (20) does not define a DF.

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