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# On the smoothness of solutions of impulsive autonomous systems

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**Abstract**

The aim of this paper is to investigate dependence of solutions on parameters for nonlinear autonomous impulsive differential equations. We will specify what continuous, differentiable and analytic dependence of solutions on parameters is, define higher order derivatives of solutions with respect to parameters and determine conditions for existence of such derivatives. The theorem of analytic dependence of solutions on parameters is proved.


*MSC:* 34A37; 34D20

*Keywords:* Autonomous impulsive differential equation; Differentiable dependence of solutions on parameters; Analytic dependence of solutions on parameters

## 1. Introduction

Differentiability of solutions of nonautonomous impulsive differential equations with respect to parameters has been considered in many works [2,3,8,16,21,23]. But smoothness of solutions of autonomous systems apparently has not yet been investigated. Since autonomous impulsive equations can define discontinuous dynamical systems [4,5,10,11,13,14,16,18–21] the subject is very important for applications. The

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investigation requires additional definitions and special methods because of its complexity. We use the topology of piecewise continuous functions which is based on ideas of [11] and was developed in [1–3]. Moreover, we shall utilize $B$-equivalence of impulsive systems [1,3]. It is the inherent advantage of the method that we consider the problem for equations with variable time of impulses, while in [8,16,23] it was discussed for systems with fixed moments of impulsive actions. Effective methods of the investigation of systems with nonfixed moments of impulsive action can be found in [7,9,16,17,21].

The paper is structured in the following way. In the next section we introduce and describe properties of some maps which will be useful in the rest of the article. The main subject of Section 3 is the continuous dependence of trajectories on parameters. Section 4 is the description of the notion of $B$-equivalent impulsive equations. The final three sections constitute the heart of the work and comprise definitions and theorems about derivatives and analytical dependence of solutions on parameters.

## 2. Preliminaries

Let $G_x \times G_\mu \subset R^n \times R^m$, where $n, m$ are fixed positive integers, be an open and bounded set. Assume that the set is the domain of the following nonlinear system:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\mu), \quad x \notin \Gamma(\mu), \\ &\Delta x|_{x\in\Gamma(\mu)} = I(x,\mu), \end{aligned} \tag{1}$$

where $\Delta x|_{t=\theta} := x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t\to\theta^+} x(t)$, $\Gamma(\mu)$ is the set where solutions perform a jump. The following assumptions will be necessary throughout the paper:

(C1) for fixed $\mu \in G_\mu$, the set $\Gamma(\mu)$ is an $n$-dimensional manifold in $R^n$ given by the equation

$$\Phi(x,\mu) = 0, \quad x \in G_x; \tag{2}$$

(C2) $f, I : G_x \times G_\mu \to R^n$, $\Phi : G_x \times G_\mu \to R^1$ and $\{f, I, \Phi\} \subset C^{(1)}(G_x \times G_\mu)$. Denote $\Phi_x = \partial\Phi/\partial x$ and let $\Phi_x f$ be the product of the matrices.

(C3)

$$\Phi_x(x,\mu) f(x,\mu) \neq 0, \quad \text{if } x \in \Gamma(\mu), \tag{3}$$

i.e., the vector field is transversal at every point of the manifold.

Below in this section we introduce functions $\tau$ and $J$ and we define properties of these functions which are very important in the remaining part of our paper. Let us consider the system of ordinary differential equations associated with (1):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\mu). \tag{4}$$

Fix $\kappa \in R$ and denote by $x(t) = x(t,\kappa,x,\mu)$ a solution of (4) and introduce a function $\tau = \tau(x,\mu)$ such that $\tau$ is the moment of the first meeting of $x(t)$ with the surface $\Gamma(\mu)$.

**Lemma 2.1.** $\tau(x,\mu) \in C^{(1)}$.

**Proof.** Differentiating $\Phi(x(\tau,\kappa,x,\mu),\mu)=0$, and using (C3) one can get that

$$\frac{\partial \Phi(x(\tau,\kappa,x,\mu),\mu)}{\partial \tau} = \frac{\partial \Phi(x(\tau,\kappa,x,\mu)}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\bigg|_{t=\tau} = \frac{\partial \Phi(x(\tau,\kappa,x,\mu)}{\partial x} f(x(\tau,\kappa,x,\mu),\mu) \neq 0.$$

The proof of the lemma follows immediately from the implicit function theorem and conditions on (4). □

**Corollary 2.1.** $\tau(x,\mu)$ *is a continuous function*.

Now let $x_1(t) = x(t,\tau,x(\tau)+I(x(\tau),\mu),\mu)$ be another solution of (4). Define a function $J(x,\mu) = x_1(\kappa) - x$. Similar to Lemma 2.1, one can show that the following assertion is true.

**Lemma 2.2.** $J \in C^{(1)}$.

The proof follows from Lemma 2.1, differentiability of $I$ and the theorem on differentiable dependence of solutions on initial data [12].

**Corollary 2.2.** $J(x,\mu)$ *is a continuous function*.

Let us consider the solution $x(t) = x(t,\kappa,x,\mu)$ of (4) again and fix $(x_0,\mu_0) \in G_x \times G_\mu$, $x_0 = (x_1^0,\ldots,x_n^0)$, $\mu_0 = (\mu_1^0,\ldots,\mu_m^0)$.

**Lemma 2.3.** *The function* $\tau(x,\mu)$ *is analytic*.

**Proof.** By applying the Cauchy theorem on the analyticity of solutions of ordinary differential equations [12] and the theorem on the analytic dependence of solutions on parameters [6], we find that the expansion

$$x(t) = \sum C_{p\alpha\ldots\lambda a\ldots l}(t-\kappa)^p (x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda (\mu_1 - \mu_1^0)^a \ldots (\mu_m - \mu_m^0)^l, \tag{5}$$

is valid if $t$ is close to $\kappa$. Then (C3) and the theorem on the analyticity of implicit functions imply that for sufficiently small $||x - x_0||$ and $||\mu - \mu_0||$ there exists a unique analytic solution of the equation $\Phi(x(\tau),\mu) = 0$. Therefore,

$$\tau = \sum E^i_{\alpha\ldots\lambda a\ldots l}(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda (\mu_1 - \mu_1^0)^a \ldots (\mu_m - \mu_m^0)^l. \tag{6}$$

The proof is complete. □

**Lemma 2.4.** *The function* $J(x,\mu)$ *is analytic*.

**Proof.** By using formula (6), the theorem on substitution of a series into a series [12] and the theorem on the analytic dependence of solutions on parameters, one can conclude that the function $J$ can be expanded in powers of the components $x - x_0$ and $\mu - \mu_0$ in a sufficiently small neighbourhood of the point $(x_0, \mu_0)$. The lemma is proved. □

## 3. The continuous dependence on the initial data and parameters

Let $x^0(t) : [t_0, T] \to R^n$, $x^0(t) = x(t, t_0, x_0, \mu_0)$, be a solution of (1). From now on we make the assumption that there exists a neighbourhood of the point $(x_0, \mu_0)$ which does not intersect the manifold $\Gamma(\mu_0)$.

Fix $\varepsilon \in R$, $\varepsilon > 0$.

**Definition 3.1** (*Akhmetov and Perestyuk [3]*). A solution $y(t) : [t_1, T_1]$ of (1) is said to be in the $\varepsilon$-neighbourhood of $x^0(t)$ if:

(1) the measure of the set $[t_0, T]\backslash[t_1, T_1] \cup [t_1, T_1]\backslash[t_0, T]$ does not exceed $\varepsilon$;

(2) every point of discontinuity of $y(t)$ lies in the $\varepsilon$-neighbourhood of a point of discontinuity of $x^0(t)$;

(3) for all $t \in [t_0, T]$ which are outside of the $\varepsilon$-neighbourhoods of the points of discontinuity of $x^0(t)$ the inequality $\|y(t) - x^0(t)\| < \varepsilon$ holds.

**Definition 3.2.** Hausdorff's topology which is built on the basis of all $\varepsilon$-neighbourhoods of piecewise solutions will be called $B$-topology.

Assume that

(K1) $\Phi(x_0, \mu_0) \neq 0$; Define $t = \theta_i$, $i = \overline{1, k}$, $t_0 < \theta_1 < \theta_2 <, \ldots, < \theta_k < T$, the intersection moments of $x^0(t)$ and $\Gamma(\mu_0)$,

(K2) the points $(x^0(\theta_i+), \mu_0)$, $i = \overline{1, k}$, are not cluster points of the manifold $\Gamma(\mu_0)$.

**Theorem 3.1.** *Assume that conditions* (C1)–(C3) *and* (K1), (K2) *are satisfied. Then the solution $x^0(t)$ depends continuously on the initial data and parameters in $B$-topology. Moreover, if $x_1(t) = x(t, t_1, \mu_1)$ is a solution of* (1) *and $(t_1, x_1, \mu_1)$ is sufficiently close to $(t_0, x_0, \mu_0)$ then $x_1(t)$ is defined for all $t \in [t_1, T]$ and meets the surface $\Gamma(\mu)$ exactly at $k$ points $t = \theta_i^1$, $i = \overline{1, k}$, $t_1 < \theta_1^1 < \cdots < \theta_1^k < T$.*

**Proof.** For a positive real number $\alpha$ we shall construct a set $G^\alpha$ in the following way. Let

$$F_\alpha = \{(t, x, \mu) | t \in [t_0, T], ||x - x^0(t)|| < \alpha, ||\mu - \mu_0|| < \alpha\},$$

$G_i(\alpha)$, $i = \overline{0, k+1}$, be, respectively, $\alpha$-neighbourhoods of points $(t_0, x_0, \mu_0)$, $(\theta_i, x(\theta_i), \mu_0)$, $i = \overline{1, k}$, $(T, x^0(T), \mu_0)$ in $R \times R^n \times R^m$, and $\bar{G}_i(\alpha)$, $i = \overline{1, k}$, be, respectively, $\alpha$-neighbourhoods of points $(\theta_i, x^0(\theta_i+), \mu_0)$, $i = \overline{1, k}$. Define $G^\alpha = F_\alpha \cup (\bigcup_{i=0}^{k+1} G_i(\alpha)) \cup (\bigcup_{i=1}^{k} \bar{G}_i(\alpha))$. Take $\alpha = h$ so small that $G^h \subset G_t \times G_x \times G_\mu$, where $G_t$ is an interval such that $[t_0, T] \subset G_t$. Fix $\varepsilon \in R$, $0 < \varepsilon < h$.

1. In view of the theorem on continuous dependence on initial data and parameters [12] there exists $\bar{\delta}_k \in R$, $0 < \bar{\delta}_k < \varepsilon$, such that $\bar{G}_k(\bar{\delta}_k) \cap \Gamma(\mu_0) = \emptyset$ and every solution $x_k(t)$ of (4) which starts in $\bar{G}_k(\bar{\delta}_k)$ is continuable to $t = T$, does not intersect $\Gamma(\mu)$ and $||x_k(t) - x^0(t)|| < \varepsilon$ for those $t$.

2. The continuity of $I$ implies that there exists $\delta_k \in R$, $0 < \delta_k < \varepsilon$, such that $(\kappa, x, \mu) \in G_k(\delta_k)$ implies $(\kappa, x + I(x, \mu), \mu) \in \bar{G}_k(\bar{\delta}_k)$.

3. Using Corollary 2.1 and continuous dependence of solutions on initial data [12], one can find $\bar{\delta}_{k-1}$, $0 < \bar{\delta}_{k-1} < \varepsilon$, such that a solution $x_{k-1}$ of (4), which starts in $\bar{G}_{k-1}(\bar{\delta}_{k-1})$, $\bar{G}_{k-1}(\bar{\delta}_{k-1}) \cap \Gamma(\mu) =$ intersects $\Gamma(\mu)$ in $G_k(\delta_k)$ (we continue the solution $x_{k-1}$ only up to the moment of the intersection), and $||x_{k-1}(t) - x^0(t)|| < \varepsilon$ for all $t$ from the common domain of $x_{k-1}(t)$ and $x^0(t)$. Continuing this process for $k-2,\ k-3, \ldots, 1$, one can obtain a sequence of families of solutions of (4) $x_i(t),\ i = \overline{1,k}$, and a number $\delta \in R$, $0 < \delta < \varepsilon$, such that if a solution $x(t) = x(t, t_0, z)$ of (1) starts in$G_0(\delta)$ then it coincides over its first interval of continuity with one of the solutions $x_1(t)$, except possibly a $\delta_1$-neighbourhood of $\theta_1$. Then, on the interval $[\theta_1, \theta_2]$ it coincides with one of the solutions $x_2(t)$, except possibly, in $\delta_1$-neighbourhoods of $\theta_1, \theta_2$, etc. Finally, one can see that the integral curve of $x(t)$ belongs to $G_\varepsilon$, it has exactly $k$ points of meetings with $\Gamma(\mu)$, $\theta_i^1$, $i = \overline{1,k}$, $|\theta_i^1 - \theta_i| < \varepsilon$, $i = \overline{1,k}$, and is continuable to $t = T$. The theorem is proved. □

## 4. $B$-equivalence

Consider the solution $x^0(t) : [t_0, T] \to R^n$ again and the following system of differential equations with fixed moments of impulse effect:

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, \mu), \quad t \neq \theta_i, \\ &\Delta y|_{t=\theta_i} = W_i(y, \mu), \end{aligned} \tag{7}$$

where the function $f$ is the same as in (1). Map $W_i$ will be defined below. Furthermore, $(\alpha, \hat{\ } \beta]$, $\{\alpha, \beta\} \subset R$, stands for an oriented interval, that is

$$(\alpha, \hat{\ }\beta] = \begin{cases} (\alpha, \beta] & \text{if } \alpha \leqslant \beta, \\ (\beta, \alpha] & \text{otherwise.} \end{cases}$$

Let $x(t, \mu)$ be a solution of (1), $x(t, \mu) = x(t, \kappa, \xi, \mu)$, and let $x(t, \mu)$ be close to $x^0(t)$ in $B$-topology by Theorem 3.1 ($\kappa$ is near $t = t_0$), and continuable to $T$. Thus, $x(t, \mu)$ has exactly $k$ discontinuity points $t = \tau_i$, $i = \overline{1,k}$, $\tau_i = \tau_i(\xi, \mu)$, $\tau_i \to \theta_i$ as $(\kappa, \xi, \mu) \to (t_0, x_0, \mu_0)$.

**Definition 4.1.** Systems (1) and (7) are said to be $B$-equivalent in $G^r$, $r \in R$, $0 < r < h$, if for every $x(t, \mu)$, whose integral curve belongs to $G^r$, system (7) admits a solution $y(t, \mu) = y(t, \kappa, \xi, \mu)$ such that

$$x(t, \mu) = y(t, \mu), \quad t \in [\kappa, T] \setminus \bigcup_{i=1}^{k} (\tau_i, \hat{\ }\theta_i]. \tag{8}$$

Particularly

$$x(\theta_i,\mu)=\begin{cases} y(\theta_i,\mu) & \text{if } \theta_i\leqslant\tau_i,\\ y(\theta_i+,\mu) & \text{otherwise,}\end{cases}$$

$$y(\tau_i,\mu)=\begin{cases} x(\tau_i,\mu), & \text{if } \theta_i\geqslant\tau_i,\\ x(\tau_i+,\mu) & \text{otherwise.}\end{cases} \tag{9}$$

Conversely, if (7) has a solution $y(t,\mu)$ and the integral curve of $y(t,\mu)$ lies in $G^r$, then there exists a solution $x(t,\mu)$ of (1) such that (8) and (9) hold.

Fix $i=\overline{1,k}$. Let $\xi(t)=x(t,\theta_i,x,\mu)$ be a solution of (4) and $\tau_i=\tau_i(\theta_i,x,\mu)$ be a meeting time of $\xi(t)$ with $\Gamma(\mu)$ and $\psi(t)=x(t,\tau_i,\xi(\tau_i)+I(\xi(\tau_i),\mu))$ is another solution of (4). Then $W_i(x,\mu)=\psi(\theta_i)-x$, or

$$\begin{aligned} W_i(x,\mu)= &\int_{\theta_i}^{\tau_i} f(\xi(s),\mu)\,\mathrm{d}s + I\left(x+\int_{\theta_i}^{\tau_i} f(\xi(s),\mu)\,\mathrm{d}s\right)\\ &+\int_{\tau_i}^{\theta_i} f(\psi(s),\mu)\,\mathrm{d}s \end{aligned} \tag{10}$$

is the map of an intersection of the plane $(\theta_i,x,\mu)$ with a neighbourhood of $(\theta_i,x^0(\theta_i),\mu_0)$ into the plane $(\theta_i,x,\mu)$.

One can see that for every fixed $i$ the function $W_i$ is a map $J$ defined in *Preliminaries* with $\kappa=\theta_i$. That is why for all $i$ the functions $W_i$ are continuously differentiable.

**Theorem 4.1.** *Systems* (1) *and* (7) *are B-equivalent. The function* $x^0(t)$ *is a solution of systems* (1) *and* (7) *simultaneously*.

**Proof.** Assume that the solution $x(t,\mu)$ of (1) is defined on $[\kappa,T]$ and it is in an $r$-neighbourhood of $x^0(t)$, $G^r\subset G_t\times G_x\times G_\mu$. Consider $y(t,\mu)$, $y(\kappa,\mu)=x(\kappa,\mu)$, a solution of (7). Denote by $\tau_i$ the discontinuity points of $x(t,\mu)$. Without loss of generality suppose that $\theta_i>\tau_i$. We shall show that (8) and (9) are valid if $W_i$ is defined in (7) by (10). By the theorem on the existence and uniqueness of solutions [12] for $t\in[\kappa,\tau_1]$, the equality $y(t,\mu)=x(t,\mu)$ is valid for all $t\in[\kappa,\tau_1]$. Particularly

$$x(\tau_1,\mu)=y(\tau_1,\mu). \tag{11}$$

Since $(\tau_1,x(\tau_1,\mu))\in G^r$ and $y(t,\mu)$ is a solution of (4) on the interval $[\tau_1,\theta_1]$, in view of continuity of the function $\tau_1(x,\mu)$, it can be calculated that $y(t,\mu)$ is in an $h$-neighbourhood of $x^0(\tau_1)$ if $r$ is sufficiently small. Furthermore, we have that

$$x(\theta_1,\mu)=x(\tau_1,\mu)+I(x(\tau_1,\mu))+\int_{\tau_1}^{\theta_1} f(x(s,\mu),\mu)\,\mathrm{d}s, \tag{12}$$

and

$$y(\theta_1+,\mu)=y(\tau_1,\mu)+\int_{\tau_1}^{\theta_1} f(y(s,\mu),\mu)\,\mathrm{d}s+W_i(y(\theta_1,\mu),\mu). \tag{13}$$

Using (10)–(13), one can see that

$$y(\theta_1+,\mu) = x(\tau_1,\mu) + \int_{\tau_1}^{\theta_1} f(y(s,\mu),\mu)\,\mathrm{d}s + \int_{\theta_1}^{\tau_1} f(y(s,\mu),\mu)\,\mathrm{d}s \\ + I(x(\tau_1,\mu)) + \int_{\tau_1}^{\theta_1} f(x(s,\mu),\mu)\,\mathrm{d}s = x(\theta_1,\mu).$$

Now, defining $x(t,\mu)$ and $y(t,\mu)$ as solutions of (4) on $(\theta_1,\tau_2]$ with a common initial value $x(\theta_1,\mu)$, one can obtain that $x(t,\mu)=y(t,\mu)$ for all $t\in(\theta_1,\tau_2]$. Continuing in the same manner for all $i=\overline{1,k}$ and $t\in[\kappa,T_0]$, we shall prove that $y(t,\mu)$ is continuable on $[\kappa,T]$ and that (8) holds. Similarly, one can show that the existence of the solution $y(t,\mu)$ on the interval $[\kappa,T]$ in the $r$-neighbourhood of $x^0(t)$ implies the existence of the solution $x(t,\mu)$ if $r$ is sufficiently small. The assertion about $x^0(t)$ is trivial. The theorem is proved. □

In this part we define derivatives of functions $\tau_i(x,\mu)$, $W_i(x,\mu)$, $i=\overline{1,k}$ at the points $(x^0(\theta_i),\mu_0)$. One should emphasize that $\tau_i$ is a map $\tau$ defined in Section 1 with $\kappa=\theta_i$. The equality $\Phi(x(\tau_i(x,\mu),\mu),\mu)=0$, implies that

$$\Phi_x f\,\mathrm{d}\tau_i + \sum_{j=1}^{n}\Phi_x\frac{\partial x^0}{\partial x_j}\,\mathrm{d}x_j + \sum_{k=1}^{m}\Phi_x\frac{\partial x^0}{\partial \mu_k}\,\mathrm{d}\mu_k + \sum_{k=1}^{m}\Phi_{\mu_k}\,\mathrm{d}\mu_k = 0.$$

Using the last expression, one can obtain that

$$\frac{\partial\tau_i}{\partial x_j} = -\frac{\Phi_x\frac{\partial x^0(\theta_i)}{\partial x_j}}{\Phi_x f},$$
$$\frac{\partial\tau_i}{\partial \mu_k} = -\frac{\Phi_x\frac{\partial x^0(\theta_i)}{\partial \mu_k}+\Phi_{\mu_k}}{\Phi_x f}. \tag{14}$$

Similarly for $W_i$ the following expressions are valid:

$$\frac{\partial W_i}{\partial x_j} = f\frac{\partial\tau_i}{\partial x_j} + \frac{\partial I}{\partial x}\left(I + f\frac{\partial\tau_i}{\partial x_j}\right) - f^+\frac{\partial\tau_i}{\partial x_j},$$
$$\frac{\partial W_i}{\partial \mu_i} = (f-f^+)\frac{\partial\tau_i}{\partial\mu_k} + \frac{\partial I}{\partial x}f\frac{\partial\tau_k}{\partial\mu_k} + \frac{\partial I}{\partial\mu_k}. \tag{15}$$

It is obvious that we can proceed to define derivatives of arbitrary order.

## 5. Differentiability of solutions in parameters

Let $x^0(t)$ be the solution of (1) which was considered above. Assume that systems (1) and (4) are $B$-equivalent in $G^r$ and there exists $\delta\in R$, $\delta>0$, such that every solution which starts in $G_0(\delta)$ is continuable to $t=T$ and has exactly $k$ points of discontinuity. Denote by $x_j(t)$, $j=\overline{1,n}$, a solution of (1), where $\mu=\mu_0$ such that $x_j(t_0)=x_0+\xi e_j=$

$(x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0)$, $\xi \in R$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let $\theta_i^j$ be the moments of discontinuity of $x_j(t)$. By Theorem 3.1, for sufficiently small $|\xi|$ the solution $x_j(t)$ is defined on $[t_0, T]$.

**Definition 5.1.** The solution $x^0(t)$ is said to be differentiable in $x_j^0$, $j = \overline{1, n}$, if

(A) there exist such constants $v_{ij}$, $i = \overline{1, k}$, such that

$$\theta_i^j - \theta_i = v_{ij}\xi + o(|\xi|); \tag{16}$$

(B) for all $t \in [t_0, T] \backslash \bigcup_{i=1}^k (\theta_i, \hat{\theta_i^j}]$, the following equality is satisfied:

$$x_j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \tag{17}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i$, $i = \overline{1, k}$.

The pair $\{u_j, \{v_{ij}\}_i\}$ is said to be a $B$-derivative of $x^0(t)$ in initial value $x_0^j$.

Let $x_0(t)$ be a solution of (1), $x_0(\kappa) = x_0$, $(\kappa, x_0, \mu_0) \in G_0(\delta)$, $\xi = \kappa - t_0$, and $t = \theta_i^0$, $i = \overline{1, k}$, be the points of discontinuity of $x_0(t)$. By Theorem 3.1, for sufficiently small $|\xi|$ the solution $x_0(t)$ is defined on $[\kappa, T]$.

**Definition 5.2.** The solution $x^0(t)$ is said to be differentiable in $t_0$ if there exists a pair $\{u_0, \{v_{i0}\}_i\}$ where $u_0(t)$ is a piecewise continuous function, with discontinuities of the first kind at points $t = \theta_i$, $i = \overline{1, k}$ and $v_{i0}$, $i = \overline{1, k}$, are real constants, such that

(A) $\theta_i^0 - \theta_i = v_{i0}\xi + o(|\xi|)$, $i = \overline{1, k}$;

(B) $x_0(t) - x^0(t) = u_0(t)\xi + o(|\xi|)$, $t \in [t_0, T] \cap [\kappa, T] \backslash \bigcup_{i=1}^k (\theta_i, \hat{\theta_i^0}]$.

The pair $\{u_0, \{v_{i0}\}_i\}$ is said to be a $B$-derivative of $x^0(t)$ in $t_0$.

Assume that $x^j(t)$, $j = \overline{1, m}$, are solutions of (1), $x^j(t_0) = x_0$, $j = \overline{1, m}$, where $\mu = \mu_0 + \xi e_j = (\mu_1^0, \mu_2^0, \ldots, \mu_{j-1}^0, \mu_j^0 + \xi, \mu_{j+1}^0, \ldots, \mu_m^0)$, $(t_0, x_0, \mu_0 + \xi e_j) \in G_0(\delta)$ and $\theta_j^i$ are the moments of discontinuity of $x^j(t)$. By Theorem 3.1, for sufficiently small $|\xi|$ the solution $x^j(t)$ is defined on $[t_0, T]$.

**Definition 5.3.** The solution $x^0(t)$ is said to be differentiable in $\mu_j$, $j = \overline{1, m}$, if there exists a pair $\{u^j, \{v_i^j\}_i\}$, where $u^j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i$, $\{v_i^j\} \subset R$, $i = \overline{1, k}$, such that

(A) $\theta_j^i - \theta_i = v_i^j \xi + o(|\xi|)$, $i = \overline{1, m}$;

(B) $x^j(t) - x^0(t) = u^j(t)\xi + o(|\xi|)$, $t \in [t_0, T] \backslash \bigcup_{i=1}^k (\theta_i, \hat{\theta_j^i}]$.

The pair $\{u^j, \{v_i^j\}_i\}$ is said to be a $B$-derivative of $x^0(t)$ in $\mu_j$.

**Lemma 5.1.** *If $f \in C^{(1)}(G_x \times G_\mu)$ and $W_i \in C^{(1)}(G_i(r))$, then the solution $x^0(t)$ of* (7) *has $B$-derivatives in the initial data and parameters.*

*Moreover,*

(1) $u_j$, $j = \overline{0, n}$, *are respectively, solutions of the linear system*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f_x(x^0(t), \mu_0)u, \quad t \neq \theta_i,$$
$$\Delta u|_{t=\theta_i} = W_{ix}(x^0(\theta_i), \mu_0)u, \tag{18}$$

*with the initial conditions* $u(t_0) = -f(x_0, \mu_0)$, $j = 0$, $u(t_0) = e_j$, $j = \overline{1, n}$, *and constants* $v_{ij} = 0$, *for all* $i, j$.

(2) $u^j$ *are solutions of the nonhomogeneous linear system*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f_x(x^0(t), \mu_0)u + \frac{\partial f(x^0(t), \mu_0)}{\partial \mu_j}, \quad t \neq \theta_i,$$
$$\Delta u|_{t=\theta_i} = W_{ix}(x^0(\theta_i), \mu_0)u + \frac{\partial W_i(x^0(\theta_i), \mu_0)}{\partial \mu_j}, \tag{19}$$

*with the initial condition* $u(t_0) = 0$, *and constants* $v_i^j = 0$ *for all* $i, j$.

**Proof.** Fix $p = \overline{1, n}$. We shall prove the lemma only for the derivative in $x_0^p$. For all other parameters the proof is analogous. Let $y_p(t) = y(t, t_0, x_0 + \xi e_p, \mu_0)$. By the theorem on differentiability with respect to parameters [12],

$$y_p(t) - x^0(t) = u_p(t)\xi + \rho(\xi), \quad \rho(\xi) = o(|\xi|), \tag{20}$$

for all $t \in [t_0, \theta_1]$. Particularly,

$$y_p(\theta_1) - x^0(\theta_1) = u_p(\theta_1)\xi + o(|\xi|). \tag{21}$$

Then

$$\begin{aligned} &y_p(\theta_1+) - x^0(\theta_1+) \\ &\quad = y_p(\theta_1) - x^0(\theta_1) + W_1(y_p(\theta_1), \mu_0) - W_1(x^0(\theta_1), \mu_0) \\ &\quad = u_p(\theta_1)\xi + \rho_1(\xi) + W_{1x}(x^0(\theta_1), \mu_0)(y_p(\theta_1) - x^0(\theta_1)) + \bar{\rho}_1(\xi) \\ &\quad = (I + W_{1x}(x^0(\theta_1), \mu_0))u_p(\theta_1)\xi + \rho_1(\xi) + W_{1x}(x^0(\theta_1), \mu_0))\rho_1(\xi) + \bar{\rho}_1(\xi). \end{aligned}$$

Since $\bar{\rho}_1 = o(|\xi|)$, we have that

$$y_p(\theta_1+) - x^0(\theta_1+) = u_p(\theta_1+)\xi + \tilde{\rho}_1(\xi), \tag{22}$$

where $\tilde{\rho}_1 = o(|\xi|)$. Denote by $U(t)$, $U(\theta_1) = I$, the fundamental matrix of solutions of the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f_x(x^0(t), \mu_0). \tag{23}$$

Again, using the theorem from [12] one can obtain that for all $t \in (\theta_1, \theta_2]$ the following relation is true:

$$\begin{aligned} y_p(t) - x^0(t) &= U(t)(y_p(\theta_1+) - x^0(\theta_1+)) + \rho(y_p(\theta_1+) - x^0(\theta_1+)) \\ &= U(t)u_p(\theta_m+)\xi + \rho_2(\xi) = u_p(t)\xi + \rho_2(\xi), \end{aligned}$$

where $\rho_2 = o(|\xi|)$. Continuing the process we can prove that (17) is valid. The formula (16) involving constants $v_i^j$ is trivial. The lemma is proved. □

**Theorem 5.1.** *Assume conditions* (C1)–(C3) *and* (K1), (K2) *to be satisfied. Then the solution* $x^0(t)$ *of* (1) *has* $B$*-derivatives in the initial data and parameters.*

*Moreover,*

(A) $u_j(t),\ j=\overline{0,n}$, *are, respectively, solutions of the equation*

$$\begin{aligned}&\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u, \quad t \neq \theta_i,\\ &\Delta u|_{t=\theta_i} = B_i u,\end{aligned} \tag{24}$$

*where*

$$A(t) = f_x(x^0(t), \mu_0), \quad B_i = (f - f+)\frac{\partial \tau}{\partial x} + \frac{\partial I}{\partial x}\left(I + f\frac{\partial \tau}{\partial x}\right),$$

*with the initial conditions* $u(t_0) = -f(x_0, \mu_0),\ j = 0,\ u(t_0) = e_j,\ j = \overline{1,n}$, *and*

$$v_{ij} = -\frac{\Phi_x u_j(\theta_i)}{\Phi_x f}, \quad v_{i0} = -\frac{\Phi_x u_0(\theta_i)}{\Phi_x f}, \quad j = \overline{1,n}, \ \ i = \overline{1,k}.$$

(B) $u^j(t),\ j = \overline{1,m}$, *are solutions of the equation*

$$\begin{aligned}&\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + g_j(t), \quad t \neq \theta_i,\\ &\Delta u|_{t=\theta_i} = B_i u + J_i,\end{aligned} \tag{25}$$

*where*

$$g_j(t) = \frac{\partial f(x^0(t), \mu_0)}{\partial \mu_j}, \quad J_j = (f - f^+)\frac{\partial \tau}{\partial \mu_j} + \frac{\partial I}{\partial x} f \frac{\partial \tau}{\partial \mu_j} + \frac{\partial I}{\partial \mu_j},$$

*with the initial condition* $u(t_0) = 0$, *and*

$$v_i^j = -\frac{\Phi_x u^j(\theta_i) + \Phi_{\mu_j}}{\Phi_x f}, \quad j = \overline{1,n}, \ \ i = \overline{1,k}.$$

The proof of the theorem follows immediately from Theorem 4.1, Lemma 5.1 and formulas (14), (15).

## 6. Higher-order derivatives

Define $\Delta t = h_0$, $\Delta x = (h_1, h_2, \ldots, h_n)$, $\Delta\mu = (h^{(1)}, h^{(2)}, \ldots, h^{(m)})$, $h = (h_0, h_1, \ldots, h_n)$. Let $\tilde{x}(t) = x(t, t_0 + \Delta t, x_0 + \Delta x, \mu_0)$ be a solution of Eq. (1), $(t_0 + \Delta t, x_0 + \Delta x, \mu_0) \in C_0(\delta)$, $t = \tau_i$, $i = \overline{1,k}$ and $\tau_i$, $i = \overline{1,k}$, be the moments of discontinuity of $\tilde{x}(t)$. By Theorem 3.1, for sufficiently small $|\Delta t|$ and $|\Delta x|$ the solution $\tilde{x}(t)$ is defined on $[t_0 + \Delta t, T]$.

**Definition 6.1.** We will say that a solution $x^0(t)$ has $B$-derivatives of up to $l$th order, inclusive, in the initial data, if there exist a piecewise continuous (with discontinuities of the first kind at the points $t = \theta_i, i = \overline{1,k}$) functions $u_{1p}, u_{2pj}, \ldots, u_{lpj\ldots s}$, and constants $v_{1p}, v_{2pj}, \ldots, v_{lpj\ldots s}$, $i = \overline{1,k}$, which continuously depend on $t_0, x_0, \mu_0$, are symmetric with respect to permutation of indices, and are such that
(A)

$$\tau_i - \theta_i = \sum_{p=0}^{n} v_{1p} h_p + \cdots + \sum_{p,j,\ldots,s=0}^{n} v_{lpj\ldots s} h_p h_j \ldots h_s + o(||h||^l); \tag{26}$$

(B) *for all points* $t \in [t_0, T] \cap [t_0 + \Delta t, T] \backslash \bigcup_{i=1}^{k} (\tau_i \hat{,} \theta_i]$ we have

$$\tilde{x}(t) - x^0(t) = \sum_{p=0}^{n} u_{1p}(t) h_p + \cdots + \sum_{p,j,\ldots,s=0}^{n} u_{lpj\ldots s} h_p h_j \ldots h_s + o(||h||^l). \tag{27}$$

We call the pairs $\{u_{1p}, \{v_{1p}\}_i\}, \ldots, \{u_{lpj\ldots s}, \{v_{lpj\ldots s}\}_i\}$ as $B$-derivatives of $x^0(t)$ in the initial data $t_0, x_i^0$, $i = \overline{1,n}$, of orders $1, 2, \ldots, l$.

**Remark 6.1.** Since we will be most interested in expansions (26), (27) in what follows, for the sake of brevity, we do not use factorials in the definition of $B$-derivatives.

Let $\bar{x}(t) = x(t, t_0, x_0, \mu_0 + \Delta\mu)$ be a solution of (1), $(t_0, x_0, \mu_0 + \Delta\mu) \in C_0(\delta)$ and $t = \tau_i, i = \overline{1,k}$, be the points of discontinuity of $\bar{x}(t)$.

**Definition 6.2.** We shall say that a solution $x^0(t)$ of (1) has $B$-derivatives of up to $l$th order, inclusive, in parameters $\mu_j$, $j = \overline{1,m}$, if there exist piecewise continuous (with discontinuities of the first kind at points $t = \theta_i, i = \overline{1,k},$) functions $u_1^p(t), u_2^{pj}, \ldots, u_l^{pj\ldots s}$ and constants $v_{1i}^p, v_{2i}^{pj}, \ldots, v_{li}^{pj\ldots s}$, $i = \overline{1,k}$, that continuously depend on $t_0, x_0, \mu_0$, are symmetric with respect to permutation of indices, and are such that
(A)

$$\tau_i - \theta_i = \sum_{p=1}^{m} v_{1i}^p h^{(p)} + \cdots + \sum_{p,j,\ldots,s=1}^{m} v_{li}^{pj\ldots s} h^{(p)} h^{(j)} \ldots h^{(s)} + o(||\Delta\mu||^l); \tag{28}$$

(B) *for all points* $t \in [t_0, T] \backslash \bigcup_{i=1}^{k} (\tau_i \hat{,} \theta_i]$ *we have that*

$$\tilde{x}(t) - x^0(t) = \sum_{p=0}^{n} u_1^p(t) h_p + \cdots + \sum_{p,j,\ldots,s=0}^{n} u_l^{pj\ldots s} h_p h_j \ldots h_s + o(||h||^l). \tag{29}$$

We call the pairs $\{u_1^p, \{v_1^p\}_i\}, \ldots, \{u_l^{pj\ldots s}, \{v_{li}^{pj\ldots s}\}_i\}$ as $B$-derivatives of $x^0(t)$ of orders $1, 2, \ldots, l$, in parameters $\mu_j$, $j = \overline{1,m}$.

**Lemma 6.1.** *If* (C1)–(C3), (K1), (K2) *hold and* $\{f, I, \Phi\} \subset C^l(G_x \times G_\mu)$, *then* $\{\theta(\tau(x,\mu)), J(x,\mu)\} \subset C^l$.

The proof of the lemma follows immediately from the existence of the implicit function $\theta(x, \mu)$ in (2) and $l$-times differentiability of the solution $x^0(t)$.

Lemma 6.1 implies that the following assertion is true.

**Lemma 6.2.** *If* (C1)–(C3),(K1),(K2) *hold and* $\{f, I, \Phi\} \subset C^l(G_x \times G_\mu)$,*then* $W_i(x, \mu) \in C^l$ *in a neighbourhood of* $(\theta_i, x^0(\theta_i), \mu_0)$, $i = \overline{1, k}$.

**Lemma 6.3.** *If* $f \in C^l(G_x \times G_\mu)$ *and* $W_i \in C^l(G_i(r))$, $i = \overline{1, k}$,*then the solution* $x^0(t)$ *of* (7) *has all* $B$*-derivatives of up to* $l$*th order inclusive in the initial data and parameters*.

The proof of the lemma is similar to the proof of the theorem of higher-order derivatives from [12] and utilizes Lemma 5.1. The following theorem is a corollary of Lemma 6.3.

**Theorem 6.1.** *Let conditions* (C1)–(C3),(K1),(K2) *hold and* $\{f, I, \Phi\} \subset C^l(G_x \times G_\mu)$. *Then the solution* $x^0(t)$ *has* $B$*-derivatives of up to* $l$*th order inclusive with respect to* $t_0, x_0^j$, $j = \overline{1, n}$, $\mu_0^i$, $i = \overline{1, m}$.

## 7. $B$-analytic dependence on parameters

Let $\hat{x}(t) = x(t, t_0, x, \mu)$ be a solution of (1), $(x, \mu) \in G_x \times G_\mu$, $x = (x_1, \ldots, x_n)$, $\mu = (\mu_1, \ldots, \mu_m)$ and $\tau_i$, $i = \overline{1, k}$, be the moments of discontinuity of the solution.

**Definition 7.1.** We shall say that the solution $\hat{x}(t)$ is $B$-analytic in $x, \mu$ in a neighbourhood of $x_0, \mu_0$, if for sufficiently small $||x - x_0||$ and $||\mu - \mu_0||$ the following representations are valid:

(A)

$$\tau_i - \theta_i = \sum D^i_{\alpha\ldots\lambda a\ldots l}(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda (\mu_1 - \mu_1^0)^a \ldots (\mu_m - \mu_m^0)^l, \tag{30}$$

$i = \overline{1, k}$, *where* $D^i_{\alpha\ldots\lambda\ldots l}$ *are real constants and the series are convergent*;

(B)

$$\hat{x}(t) - x^0(t) = \sum A_{\alpha\ldots\lambda a\ldots l}(t)(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda (\mu_1 - \mu_1^0)^a \ldots (\mu_m - \mu_m^0)^l,$$
$$for \quad t \in [t_0, T] \setminus \bigcup_{i=1}^k (\tau_i \hat{,} \theta_i], \tag{31}$$

where $A_{\alpha\ldots\lambda a\ldots l}(t)$ are piecewise continuous functions with discontinuities of the first kind at points $t = \theta_i$, $i = \overline{1, k}$, and the series is convergent for all $t \in [t_0, T] \setminus \bigcup_{i=1}^k (\tau_i \hat{,} \theta_i]$.

Let $\hat{y}(t) = y(t, t_0, x, \mu)$ be a solution of (7).

**Lemma 7.1.** *If* (C1)–(C3), (K1), (K2) *hold and* $f, W_i$ *are analytic functions in* $G_x \times G_\mu$ *then the solution* $\hat{y}(t)$ *is* $B$*-analytic in a neighbourhood of* $x_0, \mu_0$. *Moreover, the equalities* $\tau_i = \theta_i$, $i = \overline{1, k}$ *are valid.*

**Proof.** By applying the theorem on analytic dependence of solutions on parameters [6] one can obtain that for all $t \in [t_0, \theta_1]$, the following representation is valid:

$$\hat{y}(t) = \sum C_{\alpha\ldots\lambda a\ldots l}(t)(x_1 - x_1^0)^{\alpha} \ldots (x_n - x_n^0)^{\lambda}(\mu_1 - \mu_1^0) \ldots (\mu_m - \mu_m^0)^{l}, \tag{32}$$

where $C_{\alpha\ldots\lambda a\ldots l}$ are continuous functions. Owing to the analyticity of the function $W_i$, the vector $\hat{y}(\theta_1+) = \hat{y}(\theta_1) + W_i(\hat{y}(\theta_1), \mu)$ is also analytic. Therefore, if we regard $\hat{y}(t)$ on the segment $[\theta_1, \theta_2]$ as the solution of the Cauchy problem for system (4) with the initial data $(\theta_1, \hat{y}(\theta_1+))$ and apply the theorem on analytic dependence of solutions on parameters and the theorem on substitution of a series into a series, we can show that representation (32) is valid on the segment $[\theta_1, \theta_2]$. Continuing on this process one can see that (32) is valid for all $t \in [t_0, T]$. The condition (A) of Definition 7.1 is obvious. The lemma is proved. □

**Theorem 7.1.** *If* (C1)–(C3), (K1), (K2) *hold and* $f, I, \Phi$ *are analytic functions in* $G_x \times G_\mu$ *then the solution* $\hat{x}(t)$ *of the system* (1) *is* $B$*-analytic in a neighbourhood of* $x_0, \mu_0$.

**Proof.** The $B$-equivalence of (7), (1), Lemmas 2.4 and 7.1 imply that the expansion (32) for all $t \in [t_0, T] \setminus \bigcup_{i=1}^{k} (\widehat{\theta_i, \tau_i}]$ is valid. By using Lemma 2.3 it is easy to see that condition B) of Definition 7.1 is satisfied too. The theorem is proved. □

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