



Existence and Stability of Almost-Periodic Solutions of Quasi-Linear Differential Equations with Deviating Argument

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Abstract—The paper is concerned with the existence of an almost-periodic solution of the system with deviating argument

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), x(t - \tau_1(t)), \dots, x(t - \tau_k(t)), \quad (1)$$

such that the associated homogeneous linear system satisfies exponential dichotomy and deviations of the argument are not restricted by any sign assumption. The exponential stability of the solution when the system is with delay argument is considered. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

Different aspects of the theory of almost-periodic solutions of quasi-linear differential equations with deviating argument, including applications, have been investigated by many authors [1–6]. The assumption that the associated linear equation has exponential dichotomy is one of the most important for the discussion. In [2], the problem was considered for functional differential equation when a matrix of coefficients is constant or periodic with respect to time variable t , and the argument is delay. Paper [3] deals with the existence of almost-periodic solutions of a system with unique and constant deviation. The aim of the present paper is to investigate the problem for system (1), where deviations and the matrix of coefficients are almost-periodic functions. Moreover, we assume that the equation is of mixed type [7], that is, the derivative of x depends upon past as well as future values. One should emphasize that the general theory has not been considered for this type of equation as well as for equations with retarded argument or for systems of the neutral type [2,8].

Let \mathbb{N} , \mathbb{R} be sets of all natural and real numbers, respectively, and $\|\cdot\|$ be the Euclidean norm in \mathbb{R}^n , $n \in \mathbb{N}$. Let $s \in \mathbb{R}$ be a positive number. We denote $G_s = \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and $G_s^{k+1} =$

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$G_s \times G_s \cdots \times G_s$ (that is, G_s^{k+1} is k -times Cartesian product of G_s). Let a set $C_0(\mathbb{R})$ (respectively, $C_0(\mathbb{R} \times G_H^{k+1})$ for a given $H \in \mathbb{R}$, $H > 0$) be a set of all bounded and uniformly continuous on \mathbb{R} (respectively, on $\mathbb{R} \times G_H^{k+1}$) functions. For $f \in C_0(\mathbb{R})$ (respectively, $C_0(\mathbb{R} \times G_H^{k+1})$) and $\tau \in \mathbb{R}$, the translate of f by τ is the function $Q_\tau f = f(t + \tau)$, $t \in \mathbb{R}$ (respectively, $Q_\tau f(t, z) = f(t + \tau, z)$, $(t, z) \in \mathbb{R} \times G_H^{k+1}$). A number $\tau \in \mathbb{R}$ is called ϵ -translation number of a function $f \in C_0(\mathbb{R})$ ($C_0(\mathbb{R} \times G_H^{k+1})$), if $\|Q_\tau f - f\| < \epsilon$, for every $t \in \mathbb{R}$ ($(t, z) \in \mathbb{R} \times G_H^{k+1}$). A set $S \subset \mathbb{R}$ is said to be relatively dense, if there exists a number $l > 0$, such that $[a, a + l] \cap S \neq \emptyset$, for all $a \in \mathbb{R}$.

DEFINITION 1. A function $f \in C_0(\mathbb{R})(C_0(\mathbb{R} \times G_H^{k+1}))$ is called an almost-periodic (almost-periodic in t uniformly with respect to $z \in G_H^{k+1}$), if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists a relatively dense set of ϵ -translations of f .

Denote by $\mathcal{AP}(\mathbb{R})$ ($\mathcal{AP}(\mathbb{R} \times G_H^{k+1})$), the set of all such functions. The following assumptions will be needed throughout the paper.

- (C₁) $A(t) \in \mathcal{AP}(\mathbb{R})$ is an $n \times n$ matrix, $\tau_j \in \mathcal{AP}(\mathbb{R})$, $j = \overline{1, k}$.
- (C₂) $f \in \mathcal{AP}(\mathbb{R} \times G_s^{k+1})$, for every $s \in \mathbb{R}$, $s \geq 0$.
- (C₃) $\exists l \in \mathbb{R}$, $l > 0$, such that

$$\|f(t, z_1) - f(t, z_2)\| \leq l \sum_{j=0}^k \|z_1^j - z_2^j\|,$$

where $z_i = (z_i^0, \dots, z_i^k) \in \mathbb{R}^{n(k+1)}$, $i = 1, 2$.

LEMMA 1. (See [9].) If $f \in \mathcal{AP}(\mathbb{R} \times G_H^{k+1})$ and $\psi \in \mathcal{AP}(\mathbb{R})$, $\psi : \mathbb{R} \rightarrow G_H^{k+1}$, then, $f(t, \psi(t)) \in \mathcal{AP}(\mathbb{R})$ and $\text{mod}(f(t, \psi(t))) \subset \text{mod}(f, \psi)$.

LEMMA 2. If $\phi \in \mathcal{AP}(\mathbb{R})$, then, $\phi(t - \tau_j(t)) \in \mathcal{AP}(\mathbb{R})$, $j = \overline{1, k}$, and $\text{mod}(\phi(t - \tau_j(t))) \subset \text{mod}(\phi, \tau_j)$.

Lemma 2 follows immediately from Lemma 1, and since $\phi(t - x) \in \mathcal{AP}(\mathbb{R} \times G_H)$.

Denote $F_{f, \phi}(t) = f(t, \phi(t), \phi(t - \tau_1(t)), \dots, \phi(t - \tau_k(t)))$, where $\phi(t) \in C_0(\mathbb{R})$. Using Lemmas 1 and 2, it is easy to check that the following lemma is valid.

LEMMA 3. If $f \in \mathcal{AP}(\mathbb{R} \times G_H^{k+1})$ and $\phi(t) \in \mathcal{AP}(\mathbb{R})$, then, $F_{f, \phi}(t) \in \mathcal{AP}(\mathbb{R})$ and $\text{mod}(F_{f, \phi}(t)) \subset \text{mod}(f, \phi, \tau_j, j = \overline{1, k})$.

2. EXPONENTIAL DICHOTOMY AND ALMOST-PERIODIC SOLUTIONS

Let

$$\frac{dx}{dt} = A(t)x \tag{2}$$

be the associated homogeneous linear system of (1). Assume that (2) satisfies exponential dichotomy. That is, there exists a projection P and positive constants $\sigma_1, \sigma_2, K_1, K_2$, such that

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq K_1 \exp(-\sigma_1(t-s)), & t \geq s, \\ \|X(t)(I-P)X^{-1}(s)\| &\leq K_2 \exp(\sigma_2(s-t)), & t \leq s, \end{aligned} \tag{3}$$

where $X(t)$ is a fundamental matrix of (2).

Let

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s), & \text{if } t \geq s, \\ X(t)(P-I)X^{-1}(s), & \text{if } t \leq s, \end{cases}$$

be a Green's function of (2).

LEMMA 4. (See [9].) If $\psi \in \mathcal{AP}(\mathbb{R})$, then,

$$\phi(t) = \int_{-\infty}^{\infty} G(t, s) \psi(s) ds$$

is the almost-periodic function and $\text{mod}(\phi) \subset \text{mod}(A, \psi)$.

THEOREM 1. Assume that Conditions (C₁)–(C₃) are valid, equation (2) satisfies exponential dichotomy and

$$lk \left(\frac{K_1}{\sigma_1} + \frac{K_2}{\sigma_2} \right) < 1.$$

Then, there exists a unique solution $\phi(t) \in \mathcal{AP}(\mathbb{R})$ of (1) and $\text{mod}(\phi) \subset \text{mod}(A, f, \tau_j, j = \overline{1, k})$.

PROOF. Let $\Psi = \{\psi \in \mathcal{AP}(\mathbb{R}) \mid \text{mod}(\psi) \subset \text{mod}(A, f, \tau_j, j = \overline{1, k})\}$ be a complete metric space with sup-norm $\|\cdot\|_0$. Define an operator Θ on Ψ , such that

$$\Theta(\psi(t)) = \int_{-\infty}^{\infty} G(t, s) F_{f, \psi}(s) ds.$$

Lemmas 3 and 4 imply that $F_{f, \psi} \in \Psi$ and $\Theta : \Psi \rightarrow \Psi$. If $\psi_1, \psi_2 \in \Psi$, then,

$$\begin{aligned} \|\Theta(\psi_1(t)) - \Theta(\psi_2(t))\| &\leq \left\| \int_{-\infty}^t X(t) P X^{-1}(s) (F_{f, \psi_1}(s) - F_{f, \psi_2}(s)) ds \right\| \\ &\quad + \left\| \int_t^{\infty} X(t) (I - P) X^{-1}(s) (F_{f, \psi_1}(s) - F_{f, \psi_2}(s)) ds \right\| \\ &\leq lk \left(\frac{K_1}{\sigma_1} + \frac{K_2}{\sigma_2} \right) \|\psi_1(t) - \psi_2(t)\|_0. \end{aligned}$$

Thus, Θ is a contractive operator and there exists a unique almost-periodic solution of the equation

$$\psi(t) = \int_{-\infty}^{\infty} G(t, s) F_{f, \psi}(s) ds,$$

which is the solution of (1). The theorem is proved. \blacksquare

REMARK 2.1. Apparently, the general problem of existence of solutions for equations of mixed type has not been considered yet. Even for the case of advanced argument, there are certain difficulties if we try to define a solution for increasing t [10,11]. Hale remarked in [7] that “these equations seem to dictate that boundary conditions should be specified in order to obtain a solution in the way as one does for elliptic partial differential equations”. We regard the boundedness of the solution on \mathbb{R} as a boundary condition in the proof of Theorem 1. Similar arguments were used in [12] to investigate various problems for ordinary and functional differential equations. Authors of [13] used this method to prove existence of a bounded solution for the equation of advanced type when $t \geq t_0$.

3. EXPONENTIAL STABILITY

This section is concerned with the problem of exponential stability of the almost-periodic solution of the system with delay argument. Then, all solutions of system (1) are continuous. Denote by $X(t, s) = X(t)X^{-1}(s)$, the Cauchy matrix of (2). We will need the following assumptions.

(C₄) $\exists \{a, b\} \subset \mathbb{R}, b \geq 1, a > 0$, such that $\|X(t, s)\| \leq b \exp(-a(t - s)), t \geq s$.

(C₅) $blk < a$.

Assume that $\tau_j(t) \geq 0, j = \overline{1, k}$ and denote $\tau_0 = \max\{\sup_t \tau_j(t), j = \overline{1, k}\} > 0$, $m(l) = 1 - \exp(\sigma \tau_0) bl(k + 1)(a - \sigma)^{-1}$, where $\sigma \in \mathbb{R}, 0 < \sigma < a$, is fixed.

(C₆) $m(l) > 0$.

Conditions (C₁)–(C₆) and Theorem 1 imply that there exists a unique solution $\xi_0(t) \in \mathcal{AP}(\mathbb{R})$ of (1). Denote $C_0([t_0 - \tau, t_0])$ the set of all continuous on $[t_0 - \tau, t_0]$ functions and let $x(t, \phi), \phi(t) \in C_0([t_0 - \tau, t_0])$ be a solution of (1), such that $x(t, \phi) = \phi(t), t \in [t_0 - \tau, t_0]$.

DEFINITION 2. The solution $\xi_0(t)$ is called uniformly exponentially stable, if there exists a number $\sigma \in \mathbb{R}$, $\sigma > 0$, such that for every $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon)$, such that the inequality $\max_{t_0 - \tau_0 \leq t \leq t_0} \|\phi(t) - \xi_0(t)\| < \delta$ implies $\|x(t, \phi) - \xi_0(t)\| < \epsilon \exp(-\sigma(t - t_0))$, for all $t \geq t_0$.

REMARK 3.1. The definition of exponential stability [11] is slightly changed in Definition 2, so that the coefficient of exponential decay can be arbitrarily small.

THEOREM 2. Assume that (C_1) – (C_6) are valid. Then, the almost-periodic solution $\xi_0(t)$ of (1) is uniformly exponentially stable.

PROOF. One can see that $v(t) = x(t, \phi) - \xi_0(t)$ is a solution of the equation

$$\frac{dv}{dt} = A(t)v + w(t, v(t)), v(t - \tau_1(t)), \dots, v(t - \tau_k(t)), \quad (4)$$

where w satisfies

$$\|w(t, v_1) - w(t, v_2)\| \leq l \sum_{j=0}^k \|v_1^j - v_2^j\|,$$

$v_i = (v_i^0, \dots, v_i^k) \in \mathbb{R}^{n(k+1)}$, $i = 1, 2$. Thus, the problem of stability of $\xi_0(t)$ is reduced to one for the zero solution of (4). Fix $\epsilon > 0$ and denote $K(l, \delta) = (b/m(l))\delta$, where $\delta \in \mathbb{R}$, $\delta > 0$. Take δ so small, such that $K(l, \delta) < \epsilon$. Assume, without any loss of generality, that $t_0 = 0$. Let $v(t, \eta)$ be a solution of (1), such that $\eta(t) \in C_0([- \tau_0, 0])$, $\|\eta(t)\| < \delta$, $t \in [- \tau_0, 0]$. Denote Ψ_η a set of all functions, which are defined on $[- \tau_0, \infty)$ and, if $\psi \in \Psi_\eta$, then,

- 1) $\psi(t) = \eta(t)$ for $- \tau_0 \leq t \leq 0$,
- 2) $\psi(t)$ is bounded and uniformly continuous on $[0, +\infty)$, and
- 3) $\|\psi(t)\| \leq K(l, \delta) \exp(-\sigma t)$, if $t \geq 0$.

Define on Ψ_η an operator Π , such that, if $\psi \in \Psi_\eta$, then,

$$\Pi\psi = \begin{cases} \eta(t), & \text{if } - \tau_0 \leq t \leq 0, \\ X(t, 0)\eta(0) + \int_0^t X(t, s)F_{\omega, \psi}(s) ds, & \text{otherwise.} \end{cases}$$

We shall show that $\Pi : \Psi_\eta \rightarrow \Psi_\eta$. Indeed, for $t \geq 0$, it is true that

$$\begin{aligned} \|\Pi\psi\| &\leq b \exp(-at)\delta + \int_0^t b \exp(-a(t-s)) lK(l, \delta) \sum_{j=0}^{k+1} \exp(-\sigma(s - \tau_j(s))) ds \\ &\leq \exp(-\sigma t) \left[b\delta + \frac{(k+1) \exp(\sigma\tau_0) blK(l, \delta)}{a - \sigma} \right] = K(l, \delta) \exp(-\sigma t). \end{aligned}$$

Differentiating $\Pi\psi$ on $[0, \infty)$, it is easy to show that $[\Pi\psi]'$ is bounded on $[0, \infty)$ and, hence, $\Pi\psi$ is a uniformly continuous on $[0, \infty)$ function.

Let $\psi_1, \psi_2 \in \Psi_\eta$. Then,

$$\|\Pi\psi_1 - \Pi\psi_2\| \leq \int_0^t b \exp(-a(t-s)) lk \|\psi_1 - \psi_2\|_1 ds \leq \frac{blk}{a} \sup_{t \geq 0} \|\psi_1 - \psi_2\|.$$

Using a contraction mapping argument, one can conclude that there exists a unique fixed-point $v(t, \eta)$ of the operator $\Pi : \Psi_\eta \rightarrow \Psi_\eta$, which is a solution of (4). The theorem is proved. ■

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