## BRIEF COMMUNICATIONS

# ON THE EXPANSION OF SOLUTIONS TO DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDE IN A SERIES IN INITIAL DATA AND PARAMETERS

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The conditions under which the solutions of equations with discontinuous right-hand sides depend on the initial data and parameters analytically are investigated. A definition is introduced, which specifies this dependence in the case where a surface of discontinuity exists.

Let $G = G_t \times G_x \times G_\mu \subset R^1 \times R^n \times R^m$ be a bounded domain, let $G_\mu$ be a neighborhood of zero, and let $\Gamma(\mu)$, for any fixed $\mu \in G_\mu$, be a surface in $G_t \times G_x$ given by the equation $t = \tau(x, \mu)$, which splits $G$ into two subsets, $G_-$ and $G_+$, so that $G = G_- \cup \Gamma(\mu) \cup G_+$ and $t' < t''$ whenever $(t', x, \mu) \in G_-$ and $(t'', x, \mu) \in G_+$.

Consider a function $f\colon G_- \cup G_+ \to R^n$. We suppose that it is continuous on each set $G_-$ and $G_+$ and that there exists a positive number $\varepsilon$ such that the function $f$ is the restriction of certain functions $f_-(t, x, \mu)$ and $f_+(t, x, \mu)$, which are holomorphic with respect to $x$ and $\mu$ in $\varepsilon$-neighborhoods of the sets $G_-$ and $G_+$, respectively.

Given a system

$$\frac{dx}{dt} = f(t, x, \mu) \tag{1}$$

on the set $G$, we study the problem of the analytical dependence of solutions to (1) on the initial data and parameters by using the method developed in [1] for pulse systems.

Assume that, for $\mu = 0$, Eq. (1) possesses a solution $\varphi(t) = x(t, t_0, x_0, 0)$, which is determined on the segment $[t_0, T] \subset G_t$ and meets the surface $\Gamma(0)$ at the point $t = \theta$ so that

$$1 - \frac{\partial\tau}{\partial x}(\varphi(\theta), 0) f(\theta, \varphi(\theta), 0) \neq 0, \tag{2}$$

where $\partial\tau/\partial x$ is a row vector and $f$ is a column vector.

Denote by $\bar{x}(t) = x(t, t_0, x, \mu)$ a solution of system (1). Let $t = \eta$ be the time when this solution meets the surface $\Gamma(\mu)$. If $\|x - x_0\|$ and $\|\mu\|$ are sufficiently small, then $\bar{x}$ is well defined on the segment $[t_0, T]$ and the relation

$$1 - \frac{\partial\tau}{\partial x}(\bar{x}(\eta), \mu) f(\eta, \bar{x}(\eta), \mu) \neq 0 \tag{3}$$

holds. We assume that all the solutions of Eq. (1) under consideration cross the discontinuity surface at a single point.

We say that a solution $\bar{x}(t)$ depends $B$-analytically on the initial value $x$ and the parameter $\mu$ in a neighbor-

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hood of the point $(x_0, 0)$ if there exists a real number $\rho_0 > 0$ such that, under the conditions $\|x - x_0\| \leq \rho_0$ and $\|\mu\| \leq \rho_0$ the solution $\bar{x}$ can be expanded, for $t \in [t_0, T]\setminus(\theta^0, \eta]$, in powers of the components $x - x_0$ and $\mu$ with coefficients that are continuous in $x$ and $\mu$, piecewise continuous in $t$, and have a discontinuity of the first kind at the point $t = \theta$.

The difference $\eta - \theta$ is also expandable in powers of the components $x - x_0$ and $\mu$ with constant coefficients continuous in $x$ and $\mu$.

Let us establish the conditions under which the solution $\bar{x}(t)$ of system (1) is $N$-analytical in $x$ and $\mu$. We split the interval $G_t$ into the parts $G^-$, $\{\theta\}$, and $G^+$ such that $t_0 \in G^-$ and $t' < t''$ for any $t' \in G^-$ and $t'' \in G^+$.

Suppose that the function $f$ can be extended (preserving holomorphy) from the sets $G_-$ and $G_+$ to the domains $G^- \times G_x \times G_\mu$ and $G^+ \times G_x \times G_\mu$, respectively, up to the plane $t = \theta$. The function obtained as a result is denoted by $F(t, x, \mu)$. For some $\delta > 0$, the function $F$ is the restriction of the functions $F_-(t, x, \mu)$ and $F_+(t, x, \mu)$ holomorphic with respect to $x$ and $\mu$ in the $\delta$-neighborhoods of the domains $G^- \times G_x \times G_\mu$ and $G^+ \times G_x \times G_\mu$, respectively.

Suppose that there exists a neighborhood of the point $(\theta, \varphi(\theta), 0)$ (we denote it by $B$) such that the function $f$ is holomorphic with respect to $t$ in the domains $B \cap G_-$ and $B \cap G_+$.

Let us construct a mapping of the part of the plane $t = \theta$, which intersects $G$, onto itself. First, on the set $G$, we define a system

$$\frac{dx}{dt} = F(t, x, \mu) \tag{4}$$

with a discontinuous right-hand side. Then we consider the following three cases:

I. If $(\theta, x, \mu) \in G_-$, then $x^0(t)$, $x^0(\theta) = x$ is a solution of system (1) and $\xi = \xi(x, \mu)$ is the time when this solution crosses $\Gamma(\mu)$, $\xi > \theta$. In addition, let $x^1(t)$, $x^1(\xi) = x^0(\xi)$, be a solution of Eq. (4) defined on the segment $[\theta, \xi]$.

II. If $(\theta, x, \mu) \in G_+$, then we assume that $x^0(t)$, $x^0(\theta) = x$, is a solution of Eq. (4) and $\xi = \xi(x, \mu)$ is the time of intersection of this solution and the surface $\Gamma(\mu)$. In this case, by $x^1(t)$ we denote the solution of Eq. (1) with the initial condition $x^1(\xi) = x^0(\xi)$ and suppose that this solution exists on the segment $[\xi, \theta]$.

III. $(\theta, x, \mu) \in \Gamma(\mu)$. Define the mapping

$$I(x, \mu) = \begin{cases} \int_\theta^\xi f(u, x^0(u), \mu)\,du + \int_\xi^\theta F(u, x^1(u), \mu)\,du, & \text{in case I;} \\ \int_\theta^\xi F(u, x^0(u), \mu)\,du + \int_\xi^\theta f(u, x^1(u), \mu)\,du, & \text{in case II;} \\ 0, \quad \text{in case III} \end{cases}$$

and introduce the following system of differential equations with pulse influence:

$$\frac{dy}{dt} = F(t, y, \mu), \qquad t \neq \theta, \qquad \Delta y\big|_{t=0} = I(y, \mu). \tag{5}$$

By using the definition of the mapping $I(x, \mu)$, one can easily show that there exists a sufficiently small neighborhood of the trajectory of a solution $x = \varphi(t)$ in $G$ (we denote it by $\overline{G}$), where systems (1) and (5) possess the

$S$-property, i.e., if $\bar{x}(t) = x(t, t_0, x, \mu)$ and $\bar{y}(t) = y(t, t_0, x, \mu)$ are the solutions of Eqs. (1) and (5), respectively, then, for sufficiently small $\|x - x_0\|$ and $\|\mu\|$, the functions $\bar{x}$ and $\bar{y}$ take the same values on their common domain of definition with the exception of a point $t$ on $[\theta, \eta]$; here, $\eta$ is the time of intersection of the solution $\bar{x}$ and the surface $\Gamma(\mu)$; $[\theta, \eta]$ is the segment $[\theta, \eta]$ if $\theta \le \eta$ and the segment $[\eta, \theta]$ whenever $\eta < \theta$. Clearly, for sufficiently small $\|x - x_0\|$ and $\|\mu\|$, the domain of definition of the solutions $\bar{x}$ and $\bar{y}$ contains the segment $[t_0, T]$.

In what follows, systems (1) and (5) are considered in the domain $\overline{G}$.

Assume that a neighborhood $\Omega$ of the point $(x_0, 0)$ is the domain of definition of the functions $\xi$ and $I$. The following lemmas take place:

**Lemma 1.** *The function* $\xi(x, \mu)$ *is holomorphic in the domain* $\Omega$.

***Proof.*** By applying the Cauchy theorem on the holomorphy of solutions to ordinary differential equations [2] and the Poincaré theorem on the expansion of solutions in a parameter [3], we find that the expansion

$$x^0(t) = \sum C_{p\alpha\ldots\lambda a\ldots l}(t - \theta)^p (x_1 - x_1^0)^\alpha \ldots (x_n - x_n)^\lambda \mu_1^a \ldots \mu_m^l. \tag{6}$$

take place for values of $t$ close to $t = \theta$. This and condition (2), by the theorem on holomorphy of implicit functions, imply that for sufficiently small $\|x - x_0\|$ and $\|\mu\|$, there exists a unique holomorphic solution of the equation $\xi = \tau(x^0(\xi), \mu)$. Therefore,

$$\xi = \sum B_{\alpha\ldots\lambda a\ldots l}(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda \mu_1^a \ldots \mu_m^l; \tag{7}$$

hence, $\xi(x_0, 0) = \theta$. Lemma 1 is proved.

**Lemma 2.** *The function* $I(x, \mu)$ *is holomorphic in the domain* $\Omega$.

***Proof.*** By using the definition of the mapping $I$, equality (7), the theorem on substitution of a series into a series, and the Poincaré theorem, we conclude that the function $I(x, \mu) = x(\theta, \xi, x^0(\xi), \mu) - x$ can be expanded in powers of the components $x - x_0$ and $\mu$ in a sufficiently small neighborhood of the point $(x_0, \mu)$. Lemma 2 is proved.

**Lemma 3.** *A solution* $\bar{y}$ *of system* (5) *situated in a sufficiently small neighborhood of the solution* $x = \varphi(t)$ *to the generating equation can be expanded in powers of the components* $x - x_0$ *and* $\mu$.

***Proof.*** By applying the Poincaré theorem on the expansion of solutions to ordinary differential equations in powers of parameters, we obtain the representation

$$\bar{y}(t) = \sum A_{\alpha\ldots\lambda a\ldots l}(t)(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda \mu_1^a \ldots \mu_m^l \tag{8}$$

on the segment $[t_0, \theta]$, where $A$ are functions continuous in $t \in [t_0, \theta]$.

Owing to the holomorphy of the function $I(x, \mu)$, the function $\bar{y}(\theta+) = \bar{y}(\theta) + I(\bar{y}(\theta), \mu)$ can also be expanded in the series. Therefore, if we regard $\bar{y}(t)$ on the segment $(\theta, T]$ as the solution of the Cauchy problem for system (5) with the initial data $\theta$ and $\bar{y}(\theta+)$ and apply the Poincaré theorem and the theorem on substitution of a series into a series, then we show that representation (8) is valid on the segment $(\theta, T]$ as well.

Lemma 3 is proved.

The $S$-property of Eqs. (1) and (5), which has been established above, and Lemmas 1 and 3 imply that the following theorem is true:

**Theorem 1.** *The solution* $\bar{x}(t) = x(t, t_0, x, \mu)$ *of system* (1) *depends* $B$*-analytically on* $x$ *and* $\mu$ *for sufficiently small* $\|x - x_0\|$ *and* $\|\mu\|$, *i.e., for all* $t \in [t_0, T]\setminus(\theta^0, \eta]$, *the expansions*

$$\bar{x}(t) = \sum A_{\alpha\ldots\lambda a\ldots l}(t)(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda \mu_1^a \ldots \mu_m^l \tag{9}$$

*and the equality*

$$\eta - \theta = \sum D_{\alpha\ldots\lambda a\ldots l}(x_1 - x_1^0)^\alpha \ldots (x_n - x_n^0)^\lambda \mu_1^a \ldots \mu_m^l \tag{10}$$

*hold, where* $A$ *are piecewise continuous functions with discontinuities of the first kind at the point* $t = \theta$ *and* $D$ *are constants. The coefficients* $A$ *and* $D$ *continuously depend on* $x$ *and* $\mu$.

This theorem yields the following one:

**Theorem 2.** *For any fixed* $t \in [t_0, T]$, $t \neq \eta$, *the function* $\bar{x}(\bar{t}) = x(\bar{t}, t_0, x, \mu)$ *is real analytic as a function of* $x$ *and* $\mu$ *in some neighborhood of the point* $(x_0, 0)$.

Finally, we note that the differentiability of solutions of discontinuous systems with respect to the initial data of the first order was studied in [4].

## REFERENCES

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