

ASYMPTOTIC REPRESENTATION OF SOLUTIONS OF REGULARLY PERTURBED SYSTEMS
OF DIFFERENTIAL EQUATIONS WITH NONCLASSICAL RIGHT-HAND SIDE

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The problem of the B-asymptotic representation, with respect to a small parameter, of solutions of a regularly perturbed system of differential equations with impulse action on surfaces and of differential equations with discontinuous right-hand side is considered. Theorems concerning the B-analytic dependence of solutions on the small parameter are proved. Algorithms for calculating the coefficients of the expansion are developed.

In this paper, we investigate the problem of the asymptotic representation, with respect to a small parameter, of solutions of differential equations with impulse action on surfaces, and of systems of differential equations with discontinuous right-hand side. The systems considered are related to differential equations with nonclassical right-hand side, and they arise as mathematical models in the solution of important practical problems [1-4]. The results obtained here are an extension of the investigations in [5, 6].

Let $G = G_t \times G_x \times G_\mu$ be a bounded region in the space $R^1 \times R^n \times R^1$, and let G_μ be a neighborhood of zero.

In G we consider a system of differential equations with impulse action on surfaces, having the form

$$\frac{dx}{dt} = f(t, x, \mu), \quad t \neq \tau_i(x, \mu), \quad \Delta x|_{t=\tau_i(x, \mu)} = I_i(x, \mu). \quad (1)$$

We assume that the singular equation

$$\frac{dx}{dt} = f(t, x, 0), \quad t \neq \tau_i(x, 0), \quad \Delta x|_{t=\tau_i(x, 0)} = I_i(x, 0), \quad (2)$$

corresponding to (1) admits a solution $x_0(t)$ such that $x_0(t_0) = x_0$ which is defined on an interval $[t_0, T] \subset G_t$, and which has points of discontinuity $t = t_i, i = 1, p, t_0 < t_1 < t_2 < \dots < t_p < T$, such that

$$1 - \langle f(t_i, x_0(t_i), 0), \frac{\partial \tau_i(x_0(t_i), 0)}{\partial x} \rangle \neq 0, \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in R^n .

Let ε be a small positive number such that the G_ε -union of ε -neighborhoods of the points $t_i, i = 1, p$, is contained in $[t_0, T]$. We assume that

$$f \in C^{(0, k+1, k+1)}(G) \cap C^{(k, k+1, k+1)}(G_\varepsilon \times G_x \times G_\mu), \\ I_i, \tau_i \in C^{(k+1)}(G_x \times G_\mu), \quad i = \overline{1, p}.$$

In addition, we assume that conditions on the discontinuity surface hold under which there is no "beating" of the solutions of (1) [2].

In what follows, we assume that $(a, \cdot b]$ denotes the set $(a, b]$ if $a \leq b$ and the set $(b, a]$ if $b < a$. We adopt the same notation for the other forms of numerical intervals.

Let $x^0(t) = x(t, t_i, x, \mu)$ be the solution of the Cauchy problem for the equation

$$dx/dt = f(t, x, \mu) \quad (4)$$

and let $t = \xi_i$ be the time of contact of this solution with the surface $t = \tau_i(x, \mu)$. We assume that there exists a solution $x^1(t) = x(t, \xi_i, x^0(\xi_i) + I_i(x^0(\xi_i), \mu), \mu)$ of (4) on the interval $[t_i, \cdot \xi_i]$, and we define the mapping

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$$I_i(x, \mu) = \int_{t_i}^{\xi_i} f(u, x^0(u), \mu) du + I_i \left(x + \int_{t_i}^{\xi_i} f(u, x^0(u), \mu) du \right) + \int_{\xi_i}^{t_i} f(u, x^1(u), \mu) du. \quad (5)$$

It was proved in [5] that when the above conditions on (1) are fulfilled, then $I_i \in C^{(k+1)}$.

Let $x(t, \mu)$, $x(t_0, \mu) = x_0$, be a solution of (1). By virtue of the fact that the dependence of the solutions of (1) on the parameter μ is continuous in the B-topology for sufficiently small $|\mu|$ [6], this solution is defined on the entire interval $[t_0, T]$; moreover, the mapping $J_i(x, \mu)$, $i = \overline{1, p}$, is defined in some neighborhood of the point $(x_0(t), 0)$, and, by virtue of (3), the solution $x(t, \mu)$ is not tangent to any of the discontinuity surfaces. We let ξ_i , $i = \overline{1, p}$, denote the times at which the solution $x(t, \mu)$ meets the discontinuity surfaces $t = \tau_i(x, \mu)$. It is not difficult to verify that, at all points $t \in [t_0, T]$, with the possible exception of points of the intervals $(t, \xi_i]$, this solution takes on the same values as the solution $y(t, \mu)$, $y(t_0, \mu) = x_0$, of an equation of the form

$$\frac{dy}{dt} = \tilde{f}(t, y, \mu), t \neq t_i, \Delta y|_{t=t_i} = J_i(y, \mu), \quad (6)$$

with impulse actions at fixed instants of time.

We say that a solution $x(t, \mu)$ has a B-asymptotic representation if for sufficiently small $|\mu|$ and for all $t \in [t_0, T]$ outside of the intervals $[t_i, \xi_i]$,

$$x(t, \mu) = \sum_{j=0}^k x_j(t) \mu^j + O(\mu^{k+1}), \quad (7)$$

where the x_j are piecewise continuous vector-valued functions with discontinuities of the first kind at the points t_i . Moreover for all $i = \overline{1, p}$,

$$\xi_i - t_i = \sum_{j=1}^k \kappa_{ij} \mu^j + O(\mu^{k+1}), \quad (8)$$

where the κ_{ij} are real constants.

The existence of the expansions (7) and (8) is proved in [5]. We consider here the problem of determining the coefficients x_j and κ_{ij} . By virtue of the correspondence established above between the solutions $x(t, \mu)$ and $y(t, \mu)$, it suffices to determine the x_j beginning with Eq. (6), taking into account the fact that the asymptotic representation

$$y(t, \mu) = \sum_{j=0}^k x_j(t) \mu^j + O(\mu^{k+1}) \quad (9)$$

holds at all points $t \in [t_0, T]$.

Substituting (9) into (6) and using the smoothness of f and J_i , we find that $x_0(t)$ is a solution of the Cauchy problem $x_0(t_0) = x_0$ for Eq. (2). For each $j = \overline{1, k}$, the coefficient x_j is a solution of the Cauchy problem $x_j(t_0) = 0$ for an equation of the form

$$\frac{dx}{dt} = \frac{\partial f}{\partial x}(t, x_0(t), 0)x + F(t, x_0, x_1, \dots, x_{j-1}), \quad t \neq t_i, \quad (10)$$

$$\Delta x|_{t=t_i} = \frac{\partial J_i}{\partial x}(x_0(t_i), 0)x + G_i(x_0, x_1, \dots, x_{j-1}),$$

in which F and G_i are completely determined by x_0, x_1, \dots, x_{j-1} and the partial derivatives of f and J_i of order up to and including j , evaluated for $x = x_0(t)$ and $\mu = 0$.

We take up the determination of the partial derivatives of the J_i at the points $(x_0(t_i), 0)$. We fix i , and for brevity we omit the index i in what follows. We let $t_i = \eta$, and if $x = x_0(\eta)$, $t = \eta$, and $\mu = 0$, or if $x = x_0(\eta+)$, $t = \eta$, and $\mu = 0$, then all of the functions used below will be written without showing the values of the arguments, and we will distinguish between these two cases by using a + superscript in the second case. Also, we use the subscripts x, t , and μ to indicate partial derivatives. We will assume that x, f, I , and J and their derivatives are column vectors, and that the derivatives of τ and ξ are row vectors. We define the product of vectors and matrices according to the usual rule for multiplying rectangular matrices.

The point of discontinuity $t = \xi$ of the solution $x(t, \mu)$ is determined from the equation $\xi = \tau(x(\xi, \mu), \mu)$ as a function $\xi = \xi(x, \mu)$, where $x = x(\eta, \mu)$. Therefore, applying the well-known implicit function theorem and passing to the limit as $\mu \rightarrow 0$, we get

$$\begin{aligned}\xi_x &= \tau_x(1 - \tau_x f)^{-1}, \quad \xi_\mu = \tau_\mu(1 - \tau_x f)^{-1}, \\ \xi_{x\mu} &= \tau_{x\mu}(1 - \tau_x f)^{-1} + (\tau_x(\tau_{x\mu}f + \tau_x f_\mu))(1 - \tau_x f)^{-2}, \\ \xi_{\mu\mu} &= (\tau_\mu(\tau_{x\mu}f + \tau_x f_\mu))(1 - \tau_x f)^{-2} + \tau_{\mu\mu}(1 - \tau_x f)^{-1}, \\ \xi_{xxj} &= 2\tau_{xxj}(1 - \tau_x f)^{-1} + (\tau_x(2\tau_{xxj}f + \tau_x f_{xj}))(1 - \tau_x f)^{-2}.\end{aligned}\tag{11}$$

Applying the resulting expression in a similar way, starting from (5), we find

$$\begin{aligned}J_x &= \xi_x(f - f^+) + I_x(E + \xi_x f), \quad J_\mu = (f - f^+)\xi_\mu + I_x f \xi_\mu + I_\mu, \\ J_{x\mu} &= \xi_x(f_t - f_t^+)\xi_\mu + \xi_{x\mu}(f - f^+) + (f_x - f_x^+(E + I_x))\xi_\mu + \\ &\quad + \left(\sum_{i=1}^n I_{xx_i} f_i \xi_\mu + I_{x\mu}\right)(E + \xi_x f) + I_x(\xi_x f_\mu + f_x \xi_\mu + \xi_{x\mu} f), \\ J_{xx_i} &= \xi_x((f_t - f_t^+)\xi_{x_j} + f_{x_j} - f_{x_j}^+) + \xi_{xx_j}(f - f^+) + \xi_{x_j}(f_x - f_x^+ + f_x^+ I_x) + \\ &\quad + \sum_{i=1}^n I_{xx_i}(\delta_{ij} + f_i \xi_{x_j})(E + \xi_x f) + I_x(\xi_x(f_{x_j} + f_i \xi_{x_j})) + \xi_{xx_j} f + \xi_{x_j} f_x, \\ J_{\mu\mu} &= (f_t - f_t^+)\xi_\mu^2 + (f_\mu - f_\mu^+)\xi_{\mu\mu} + (f_\mu + I_{x\mu} f)\xi_\mu + \left(\sum_{i=1}^n I_{xx_i} f_i \xi_\mu\right)(f_\mu - I_\mu) + \\ &\quad + I_x(f_t \xi_\mu^2 + 2f_\mu \xi_\mu + f_\mu \xi_{\mu\mu}) + \sum_{i=1}^n I_{\mu x_i} f_i + I_{\mu\mu},\end{aligned}\tag{12}$$

where δ_{ij} is the Kronecker symbol.

It is clear that in this way it is possible to calculate the derivatives of J up to and including order k at the point $(x_0(\eta), 0)$ from the values of the derivatives of f , I , and τ at the points $(\eta, x_0(\eta), 0)$ and $(\eta, x_0(\eta+), 0)$. In addition, the coefficients in (8) can be determined by starting from (11).

THEOREM 1. Suppose that the system (1) and the solution $x_0(t)$ of the singular system (2) satisfy the above conditions. Then for $|\mu|$ sufficiently small the solution $x(t, \mu)$, $x(t_0, \mu) = x_0$ is defined on the interval $[t_0, T]$ and admits a B-asymptotic representation there. The coefficients x_j in (7) can be determined by recursion as solutions of the linear nonhomogeneous system (10) for $j = 1, k$ with the help of expressions of the form (11) and (12). The constants κ_{ij} in (8) are equal to the values of the function $(1/j!)(\partial^j \xi_i / \partial \mu^j)$ at the point $(x_0(t_j), 0)$ and can be calculated by means of a formula of the form (11).

We consider now the system (1), assuming that the functions f , I , τ are holomorphic in x and μ in the region G , and that, in addition, f is holomorphic in t , x , and μ in the region $G_\xi \times G_x \times G_\mu$. We also assume that

$$\|f\| \leq M < +\infty, \quad \left\| \frac{\partial \tau}{\partial x} \right\| \leq C < +\infty,\tag{13}$$

uniformly in $(t, x, \mu) \in G$, that

$$MC < 1\tag{14}$$

and that, for all i, x, μ and $\tau_i(x, \mu) \geq \tau_i(x + I_i(x, \mu), \mu)$. Then each solution of the system (1) can meet an arbitrary discontinuity surface not more than once [2]. From the results in [5] it follows that if the above conditions are fulfilled for sufficiently small $|\mu|$ the solution $x(t, \mu)$ depends B-analytically on μ , i.e., there exist piecewise continuous vector-valued functions x_j , $j = 1, 2, \dots$, with discontinuities of the first kind at the points t_j , and there exists a sequence of real numbers $\{x_{ij}\}_j$, $i = 1, p$, such that 1) for all $t \in [t_0, T]$ outside of the intervals $(t_i, \xi_i]$,

$$x(t, \mu) = \sum_{j=0}^{\infty} x_j(t, \mu)^j\tag{15}$$

2) for all $i = \overline{1, p}$,

$$\xi_i - t_i = \sum_{j=1}^{\infty} x_{ij} \mu^j. \quad (16)$$

Substituting (15) into (6) and successively equating terms of the same degree in μ , we can see that the following theorem holds.

THEOREM 2. Suppose that the system (1) and the solution $x_0(t)$ of the singular equation (2) satisfy the above conditions. Then the solution $x(t, \mu)$, $x(t_0, \mu) = x_0$, of the system (1) depends B-analytically on μ in a neighborhood of the function $x_0(t)$, and the coefficients x_j in (15) can be determined from the linear nonhomogeneous equations (10), $j = 1, 2, 3, \dots$, with the help of expressions of the form (11) and (12). The constants x_{ij} are equal to the values of the functions $(1/j!)(\partial^j \xi_i / \partial \mu^j)$ at the points $(x_0(t_i), 0)$.

Next we study the problem of the asymptotic representation of the solution of a system of differential equations with discontinuous right-hand side. For this, we apply the above methodology for impulse systems.

The differential properties of the solutions of equations with discontinuous right-hand side have been studied in [3, 4]. In [3] variational equations are identified which are satisfied by the principal part of the difference of two close solutions in cases where the solutions cross a surface of discontinuity, are contained in it, or converge to it. It is proved there that, in general, the investigations of the differential properties of the solutions for the above cases do not differ. Therefore, we will consider the problem assuming that the solutions cross the discontinuity surface.

Suppose that the system

$$dx/dt = f(t, x, \mu), \quad (17)$$

is given in the region G , where the function f has discontinuities on the surface $\Gamma(\mu)$ given by the equation $t = \tau(x, \mu)$. We suppose that the solution $x_0(t)$, $x_0(t_0) = x_0$ of the corresponding singular equation

$$dx/dt = f(t, x, 0), \quad (18)$$

exists on the interval $[t_0, T]$ and intersects the surface $\Gamma(0)$ at the point t_1 , $t_0 < t_1 < T$, so that the relation

$$1 - \left(f(t_1 \pm, x_0(t_1), 0), \frac{\partial \tau(x_0(t_1), 0)}{\partial x} \right) \neq 0, \quad (19)$$

involving the limiting values of f , holds. We also assume that for each fixed μ the surface $\Gamma(\mu)$ splits the set G into two regions, $G_+(\mu)$ and $G_-(\mu)$. For definiteness, we will assume that the point $(t_0, x_0, 0)$ belongs to $G(0)$, which means that, for sufficiently small $|\mu|$, (t_0, x_0, μ) belongs to $G_-(\mu)$. Let G_0 be some neighborhood of the surface $\Gamma(0)$ in the set G . We assume that, for each fixed μ ,

$$f \in C^{(0, k+1, k+1)}(G_+ \cup G_-) \cap C^{(k, k+1, k+1)}((G_+ \cup G_-) \cap G_0), \\ \tau \in C^{(k+1)}(G),$$

and that f and each of its partial derivatives have finite limits at points of Γ .

We make a construction which will be useful in what follows. We split the interval G_t into two nonintersecting parts, G_t^- and G_t^+ , containing the points $t < t_1$ and $t > t_1$, respectively. We extend f and all of its partial derivatives continuously from $G_+ \cap (G_t^+ \times G_x \times G_\mu)$ to $G_t^+ \times G_x \times G_\mu$, and from $G_- \cap (G_t^- \times G_x \times G_\mu)$ to $G_t^- \times G_x \times G_\mu$ so that they have finite limits at each point of the plane $t = t_1$. We denote the extension of f by f_1 .

Now let $x^0(t, \mu) = x(t, t_1, x, \mu)$ be a solution of the Cauchy problem for Eq. (17), let $t = \xi$ be the time of its contact with the surface $\Gamma(\mu)$, and let $x^1(t, \mu) = x(t, \xi, x^0(\xi, \mu), \mu)$ be a solution of the equation

$$dx/dt = f_1(t, x, \mu), \quad (20)$$

defined on $[t_1, \cdot \xi]$. We construct the mapping

$$J(x, \mu) = \int_{t_1}^{\xi} f(u, x^0(u, \mu), \mu) du + \int_{\xi}^{t_1} f(u, x^1(u, \mu), \mu) du.$$

Let $x(t, \mu)$, $x(t_0, \mu) = x_0$, be a solution of (17), and let $t = \xi_1$ be the time of contact of this solution with $\Gamma(\mu)$. It is not difficult to verify that the solution $x(t, \mu)$, which, for sufficiently small $|\mu|$, exists on $[t_0, T]$ and outside of $[t_1, \cdot \xi_1]$ takes on the same values as the solution $y(t, \mu)$, $y(t_0, \mu) = x_0$, of the impulse system

$$\frac{dy}{dt} = f_1(t, y, \mu), \quad t \neq t_1, \quad \Delta y|_{t=t_1} = J(y, \mu). \quad (21)$$

Since it is possible to apply to (21) all of the arguments applied to (6), it follows on the basis of the proof of Theorem 1 that, for all $t \in [t_0, T]$ outside of the interval $[t_1, \cdot \xi_1]$, an equality of the form (7) is true, and (8) holds for $i = 1$. Moreover, $x_0(t)$ is the solution of (18) with initial condition $x_0(t_0) = x_0$, and each x_j , $j = 1, k$, is a solution of a system of the form

$$\frac{dx}{dt} = \frac{\partial f}{\partial x}(t, x_0(t), 0)x + F(t, x_0, x_1, \dots, x_{j-1}), \quad t \neq t_1, \quad (22)$$

$$\Delta x|_{t=t_1} = \frac{\partial J}{\partial x}(x_0(t_1), 0)x + G(x_0, x_1, \dots, x_{j-1}).$$

We find the partial derivatives of $J(x, \mu)$ used in the construction of the system (22) by starting from expressions of the form (11) and (12) and putting $I(x, \mu) \equiv 0$; the function f and its partial derivatives have only limiting values at the point $(t_1, x_0(t_1), 0)$ as $\mu \rightarrow 0$. Then for these derivatives up to the second order, we have

$$\begin{aligned} \xi_x &= \tau_x(1 - \tau_x f^-)^{-1}, \quad \xi_\mu = \tau_\mu(1 - \tau_x f^-)^{-1}, \\ \xi_{x\mu} &= \tau_{x\mu}(1 - \tau_x f^-)^{-1} + (\tau_x(\tau_{x\mu} f^- + \tau_x f^-)) (1 - \tau_x f^-)^{-2}, \\ \xi_{xx_j} &= 2\tau_{xx_j}(1 - \tau_x f^-)^{-1} + (\tau_x(2\tau_{xx_j} f^- + \tau_x f^-)) (1 - \tau_x f^-)^{-2}, \\ \xi_{\mu\mu} &= (\tau_\mu(\tau_{\mu\mu} f^- + \tau_\mu f^-)) (1 - \tau_x f^-)^{-2} + \tau_{\mu\mu}(1 - \tau_x f^-)^{-1} \end{aligned}$$

and

$$\begin{aligned} J_x &= \xi_x(f^- - f^+), \quad J_\mu = (f^- - f^+) \xi_\mu, \\ J_{xx_j} &= \xi_x[(f_i^- - f_i^+) \xi_{x_j} + f_{x_j}^- - f_{x_j}^+] + \xi_{xx_j}(f^- - f^+) + \xi_{x_j}(f_x^- - f_x^+), \\ J_{x\mu} &= \xi_x(f_i^- - f_i^+) \xi_\mu + \xi_{x\mu}(f^- - f^+) + (f_x^- - f_x^+) \xi_\mu, \\ J_{\mu\mu} &= (f_i^- - f_i^+) \xi_\mu^2 + (f^- - f^+) \xi_{\mu\mu} + f_\mu \xi_\mu. \end{aligned}$$

Hence the following theorem holds.

THEOREM 3. Suppose that the system (17) satisfies the above conditions. Then, for $|\mu|$ sufficiently small, the solution $x(t, \mu)$, $x(t_0, \mu) = x_0$, of (17) exists on $[t_0, T]$ and admits a B-asymptotic representation given by (7) and (8) (for $i = 1$). The coefficients x_j in (7) can be determined by recursion as solutions of (23), $j = 1, k$, and the constants κ_{1j} are equal to the values of the functions $(1/j!)(\partial^j \xi / \partial \mu^j)$ at the point $(x_0(t_1), 0)$.

The proof of the following theorem is similar to that of Theorem 2.

THEOREM 4. Suppose that in (17) the function f is holomorphic in x and μ in the region $G_+ \cup G_-$, and holomorphic in t in the region $(G_+ \cup G_-) \cap G_0$, and that the function τ is holomorphic in $G_+ \cup G_-$. Suppose also that $G_+ \cup G_-$ and $\|f\| \leq M$, $\|\partial \tau / \partial x\| \leq C$, uniformly in $MC < 1$. The solution $x_0(t)$ of the singular equation (18) which satisfies the initial condition $x_0 \times (t_0) = x_0$ exists on $[t_0, T]$ and satisfies (19).

Then for $|\mu|$ sufficiently small, the solution $x(t, \mu)$, $x(t_0, \mu) = x_0$, of (17) exists on $t \in [t_0, T]$, and for all t in $[t_0, T]$ except those in $[t_1, \cdot \xi_1]$ (15) holds and (16) holds for $i = 1$. The coefficients x_j can be determined by recursion as solutions of (23), $j = 1, 2, \dots$, and the constants κ_{1i} are equal to the values of $(1/j!)(\partial^j \xi / \partial \mu^j)$ at $(x_0(t_1), 0)$.

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OPTIMAL RECONSTRUCTION OF CONVOLUTIONS AND SCALAR PRODUCTS
OF FUNCTIONS FROM VARIOUS CLASSES

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There is investigated the problem of optimal reconstruction of convolutions and scalar products of functions from various functional classes, with respect to an optimal linear information about these functions.

Let $L_p, 1 \leq p \leq \infty$ be the space of real 2π -periodic functions with the corresponding norms $\|\cdot\|_p, \mathfrak{M}_1, \mathfrak{M}_2 \subset L_2, f_1 \in \mathfrak{M}_1, f_2 \in \mathfrak{M}_2$, and let

$$(f_1 * f_2)_{(t)} = \int_0^{2\pi} f_1(x-t) f_2(t) dt$$

be the convolution of functions f_1 and f_2 , and

$$(f_1, f_2) = \int_0^{2\pi} f_1(t) f_2(t) dt$$

be its scalar product. Further, suppose that on linear hulls $\text{lin}(\mathfrak{M}_l), l=1,2$ of sets \mathfrak{M}_l there are given sets $T_l = (T_{l,1}, \dots, T_{l,n_l})$ of linear continuous functionals $T_{l,j}: \text{lin}(\mathfrak{M}_l) \rightarrow \mathbb{R}, j=1, \dots, n_l$. Vectors $T_l(f_l) = (T_{l,1}(f_l), \dots, T_{l,n_l}(f_l))$ will be called the linear information about f_1 and f_2 . An arbitrary function $\Phi = \Phi(x, y, t), x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}, t \in \mathbb{R}$ will be called a method of the reconstruction of the convolution $f_1 * f_2$, with respect to a preassigned information. The quantity

$$R(\mathfrak{M}_1, \mathfrak{M}_2; T_1, T_2; \Phi) = \sup_{\substack{f_1 \in \mathfrak{M}_1 \\ f_2 \in \mathfrak{M}_2}} \|(f_1 * f_2)_{(t)} - \Phi(T_1(f_1), T_2(f_2); \cdot)\|_\infty$$

will be called the error of the reconstruction of the convolution $f_1 * f_2$ on classes $\mathfrak{M}_1, \mathfrak{M}_2$, with respect to an information $T_1(f_1), T_2(f_2)$, with the help of a given method Φ . The quantity

$$R_n(\mathfrak{M}_1, \mathfrak{M}_2) = \inf_{T_1, T_2} \inf_{\Phi} R(\mathfrak{M}_1, \mathfrak{M}_2; T_1, T_2; \Phi) \tag{1}$$

will be called the optimal error of the reconstruction of the convolution $f_1 * f_2$ on classes \mathfrak{M}_1 and \mathfrak{M}_2 , with respect to a linear information f_1 and f_2 . The sets $T_1^* = (T_{1,1}^*, \dots, T_{1,n}^*), T_2^* = (T_{2,1}^*, \dots, T_{2,n}^*)$, and the function Φ^* , realizing the infima in the right side of (1) (if, of course, they exist), are called, respectively, the optimal information and the optimal method of reconstruction.

An arbitrary real function $G = G(x, y), x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}$ will be called a method of reconstruction of a scalar product (f_1, f_2) with respect to a preassigned information. The quantity