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ORIGINAL ARTICLE

A Hopfield neural network with multi-compartmental activation

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Abstract The Hopfield network is a form of recurrent artificial neural network. To satisfy demands of artificial neural networks and brain activity, the networks are needed to be modified in different ways. Accordingly, it is the first time that, in our paper, a Hopfield neural network with piecewise constant argument of generalized type and constant delay is considered. To insert both types of the arguments, a multi-compartmental activation function is utilized. For the analysis of the problem, we have applied the results for newly developed differential equations with piecewise constant argument of generalized type beside methods for differential equations and functional differential equations. In the paper, we obtained sufficient conditions for the existence of an equilibrium as well as its global exponential stability. The main instruments of investigation are Lyapunov functionals and linear matrix inequality method. Two examples with simulations are given to illustrate our solutions as well as global exponential stability.

Keywords Hopfield neural networks · Equilibrium · Exponential stability · Piecewise constant argument of generalized type · Constant delay · Linear matrix inequality

1 Introduction

Hopfield neural network [1] has been a popular research topic since Hopfield proposed it. The qualitative properties of this model such as exponential stability, periodicity, almost periodicity, domain of attraction and convergence rate have been deeply investigated in the papers [2–10] and references therein. The model can be expressed by the following equations:

$$C_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t)) + I_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where C_i is the input capacitance of the cell membrane, R_i is the transmembrane resistance, T_{ij} represents the connection strength between the neurons i and j, $u_i(t)$ stands for the state vector of the ith unit at time t, $g_j(u_j(t))$ denote the activation function corresponding to the unit j at time t and I_i is the external constant input to neuron i.

Hopfield neural networks have been evolved by adding impulses and delays [11–18]. Time delays often come upon in various types of systems such as mechanical systems, population models, neural networks. They affect the qualitative properties of the systems such as stability and oscillation. In neural systems, they can occur during propagation of the action potential along the axon or transmission of the electrical signal across the synapse [19–21]. So time delays may depend on conduction velocity, axon length, membrane structure and chemical kinetics.

Marcus and Westervelt introduced a single delay into (1) [22]. They analyze the dynamics of continuous-time analog networks with delay and consider the following system:

$$C_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau)) + I_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

Up to now fixed time delays, time-varying delays and distributed time delays have been added to Hopfield neural

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network models and examined by many authors [23–30]. They give some conditions ensuring existence, uniqueness, and global asymptotic stability or global exponential stability of the equilibrium point of Hopfield neural network models with delays. Besides Hopfield neural networks, Cohen–Grossberg neural networks and Bidirectional Associative Memory with delay are widespread research topics [31–41].

Differential equations with piecewise constant argument have been under investigation for many years [42]. In many real-world problems such as mechanical and biological systems, some actions on the systems can be considered as piecewise constants [43–46]. Also piecewise constant argument represents both difference and differential equations. The concept of differential equations with piecewise constant argument of generalized type (EPCAG) is introduced in [43]. It was developed in the papers [47–54]. There are many interesting results and applications of this theory in [44]. Let us now consider reasons for the involvement of piecewise constant argument. It is important that piecewise constant argument is a deviated one [43]. It is seen, if we present $\gamma(t) = t - [t - \gamma(t)]$. Then $t - \gamma(t)$ is a deviation from t. Moreover, it is seen in Fig. 1 that deviated argument of alternating type that is delayed and advanced. This is why the biological reasons which are mentioned for delays inherit for the piecewise constant argument because of its specificity. One can give more reasons to apply piecewise constant argument such as the discreteness of chemical processes in neurons (This discreteness relates to the threshold-type dynamics in neural activity) and sigmoid-type activation function in the models.

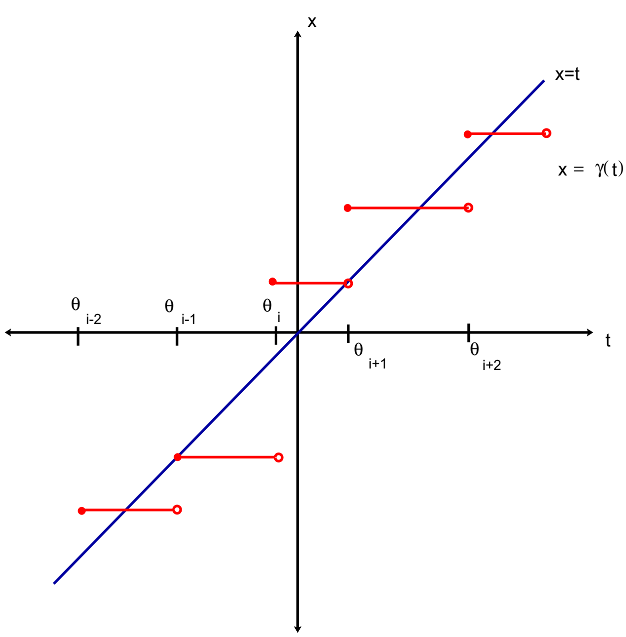

Fig. 1 The graph of the $\gamma(t)$ function

In [42], recurrent neural networks with piecewise constant argument are firstly introduced. Sufficient conditions are obtained for global exponential stability of the equilibrium point by using Lyapunov function technique for the following model:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))) + I_i, \quad i = 1, \ldots, n. \tag{3}$$

In this paper, we examine the Hopfield neural networks with both the delay and piecewise constant argument of generalized type. To the best of our knowledge, this is the first time that Hopfield neural networks with a constant delay and piecewise constant argument of generalized type are considered. In the model of neural networks, we may assume that the values of the voltage can not only be evaluated by a neural network continuously but fixed in some discrete moments of time. This may occur in the evolution of the brain during the practical activity.

We also use Lyapunov function theory to get the stability criteria in terms of LMIs. The LMI method is an efficient and popular method for studying for the stability of neural networks [14, 41, 55–58]. In [29, 30], LMI method was considered with several constant delays. In this paper, we consider a different case when activation depends on a constant delay and piecewise constant argument. So we extended the LMI technique and developed the method for a more complex situation. In our LMI, there is a 3×3 matrix and its corresponding vectors including the activation function with time t, time delay and the piecewise constant argument. This multi-compartmental structure in the inequality may facilitate the analysis of the qualitative properties of the more complex neural network systems.

Let N and R^+ denote the sets of natural and nonnegative real numbers, respectively, i.e., $N = \{0, 1, 2, \ldots\}$, $R^+ = [0, \infty)$, R^n denotes the n dimensional real space. The notation $X > 0$ (or $X < 0$) means that X is a symmetric and positive definite (or negative definite) matrix. For real symmetric matrices X and Y, the notation $X \neq Y$ (respectively, $X > Y$) means that the matrix XY is positive semi-definite (respectively, positive definite). The notations X^T and X^{-1} represent, respectively, the transpose and the inverse of a square matrix X. $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ represent the maximal eigenvalue and minimal eigenvalue of X, respectively. The norm $\|\cdot\|$ means either one-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, $x \in R^n$ or the induced matrix 2-norm: $\|X\|_2 = \sqrt{\lambda_{\max}(X^T X)}$. Let θ_i, and ζ_i, denote two fixed real-valued sequences such that $\theta_i < \theta_{i+1}$, $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in N$, with $\theta_i \to \infty$ as $i \to \infty$. Throughout the paper, we assume that there exists a positive constant $\bar{\theta}$ such that $\theta_{i+1} - \theta_i \leq \bar{\theta},\ i \in N$.

Consider the description of the following neural network with piecewise constant argument of generalized type and constant delay:

$$\begin{aligned} x_i'(t) = &-a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) \\ &+ \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))) \\ &+ \sum_{j=1}^{n} d_{ij} g_j(x_j(t-\tau)) + E_i, \\ &i = 1, \ldots, n, \end{aligned} \tag{4}$$

where $\gamma(t) = \zeta_k$ if $t \in [\theta_k, \theta_{k+1}),\ k \in N,\ t \in R^+$, $x_i(t)$ is the neuron state vector, $g_j(x_j(t))$ is the activation function of neuron j at time t and $E = [E_1, \ldots, E_n]^T$ is an external constant input vector.

Additionally, we have $a_i > 0$ and $b_{ij},\ c_{ij},\ d_{ij}$ are real constants, denote the connection weights. The equations in (4) have the *three compartmental activation* with coefficients $b_{ij},\ c_{ij}$ and d_{ij} respectively. In the first compartment, the activation utilizes values of x at the present time t. In the second, the argument is piecewise constant. In the third one, it is delayed.

(A1) The activation functions g_i satisfy $g_i(0) = 0$ for each $i = 1, 2, \ldots, n$;

(A2) There exists Lipschitz constant $L = \mathrm{diag}(L_1, \ldots, L_n) > 0$, such that $|g_i(u) - g_i(v)| \leq L_i|u - v|$, for all $u, v \in R^n,\ i = 1, 2, \ldots, n$;

(A3) The activation function g_i is bounded, i.e., for some constant $M_i > 0$, $|g_i(x_i(t))| < M_i$, for all $t \in R,\ x \in R$ and $i = 1, 2, \ldots, n$;

(A4) $\bar{\theta} < \tau$.

Our brain consists of billions of cells. Each of these cells behaves as a complex system in itself. Every cell continually sends electrical signals to other cells, and there are tens of thousands of connections between these cells. Building a model which represents the complex connections between the neurons may not be possible. But the main aim is to make the closest approach to the human brain structure. Because of the numerous numbers of cells and variety of connections, we have to say about multi-compartmental structure. We consider each sum which is located on the right-hand side of the main equation as a compartment. The compartments differ from each other since the different types of arguments in $x(.)$. Thus, in our paper the multi-compartmental activation consists of the three sums. The activation function is used to determine the multi-compartmental activation. We apply this concept of multi-compartmental model to describe the variety of types of arguments in activation functions in a short way. The number of the compartments can be increased to any natural.

System (4) is a developed version of the Hopfield model by introducing a piecewise constant argument as well as a constant delay. Reasons for the development have been stated in [47–50]. In a word, piecewise constant argument combines the continuous and discrete dynamical systems. One can suggest to investigate more general equations with functional dependence on the piecewise constant argument for the Hopfield model [1]. But this generalization makes the analysis far from the applications [1].

The assumptions $(A1)$, $(A2)$ and $(A3)$ guarantee the existence and uniqueness of equilibrium point of the system (4) [54]. Denote the equilibrium point, $x^* = (x_1^*, \ldots, x_n^*)^T$, of the system (4). Now consider the following initial value problem

$$\begin{aligned} x_i'(t) = &-a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) \\ &+ \sum_{j=1}^{n} c_{ij} g_j(x_j(\gamma(t))) \\ &+ \sum_{j=1}^{n} d_{ij} g_j(x_j(t-\tau)) + E_i, \end{aligned} \tag{5}$$

$$x_i(t) = \varphi_i(t),\ \ \sigma - \tau \leq t \leq \sigma, \tag{6}$$

for $i = 1, \ldots, n$, where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1}),\ k \in N,\ t \in R^+$ and $\varphi_i(t)$ is a continuous function.

The initial value problem (5), (6) admits a unique solution $x(t, \sigma, \varphi)$ on interval $[\sigma, \zeta_i]$ under the assumptions (A1)–(A4). The initial moment, σ, is such that $\theta_i \leq \sigma < \zeta_i < \theta_{i+1}$, for some $i \in \mathbb{Z}$ [50].

Definition 1 The equilibrium $x = x^*$ of (4) is said to be globally exponentially stable if there exist positive constants K and α such that

$$||x(t) - x^*|| \leq K e^{\alpha t} \sup_{-\tau \leq \xi \leq 0} ||x(\xi) - x^*||.$$

We will substitute $x(t) = u(t) + x^*$, then the system (4) can be simplified as

$$\begin{aligned} u_i'(t) = &-a_i u_i(t) + \sum_{j=1}^{n} b_{ij} G_j(u_j(t)) \\ &+ \sum_{j=1}^{n} c_{ij} G_j(u_j(\gamma(t))) \\ &+ \sum_{j=1}^{n} d_{ij} G_j(u_j(t-\tau)), \\ &i = 1, \ldots, n, \end{aligned} \tag{7}$$

where $G_j(u_j(t)) = g_j(u_j(t) + x_j^*) - g_j(x_j^*)$, with $g_j(0) = 0$.

A trivial verification shows that the stability of the zero solution of (7) is equal to that of the equilibrium x^* of (4). Therefore, we take into account the stability of the zero solution of (7).

Lemma 1 [59] *Given any real matrices* U_1, U_2, W *of appropriate dimensions and a scalar* $\epsilon > 0$ *such that* $0 < W = W^T$, *then the following matrix inequality is true*:

$$U_1^T U_2 + U_2^T U_1 \le \epsilon U_1^T W U_1 + \frac{1}{\epsilon} U_2^T W^{-1} U_2.$$

2 Main results

Theorem 1 Let assumptions (A1)–(A4) hold. The equilibrium x^* of (7) is globally exponentially stable, if there exist matrices $P > 0$, $Q > 0$ and two diagonal matrices $R > 0$, $S > 0$ such that

$$\begin{pmatrix} \Omega & -PC & -PD \\ -C^T P & S & 0 \\ -D^T P & 0 & Q \end{pmatrix} > 0, \tag{8}$$

where $\Omega = AP + PA - PBRB^T P - L(R^{-1} + Q + S)L$, $A = \mathrm{diag}(a_1, \ldots, a_n)$; $a_i > 0$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$ are connection weight matrices.

Proof Firstly, we choose a functional candidate for the system (7) as below

$$V(u_t) = u^T(t)Pu(t) + \int_{t-\tau}^{t} G^T(u(\xi))QG(u(\xi))\mathrm{d}\xi + \int_{\gamma(t)}^{t} G^T(u(\xi))SG(u(\xi))\mathrm{d}\xi.$$

Then, we will find the time derivative of $V(u_t)$ along the trajectories of system (7)

$$\begin{aligned} \dot V(u_t) = & -u^T(t)(A^T P + PA)u(t) + G^T(u(t))B^T Pu(t) \\ & + G^T(u(\gamma(t)))C^T Pu(t) + G(u(t-\tau))D^T Pu(t) \\ & + u^T(t)PBG(u(t)) + u^T(t)PCG(u(\gamma(t))) \\ & + u^T(t)PDG(u(t-\tau)) + G^T(u(t))QG(u(t)) \\ & - G^T(u(t-\tau))QG(u(t-\tau)) + G^T(u(t))SG(u(t)). \end{aligned} \tag{9}$$

It follows from Lemma (1),

$$\begin{aligned} & u^T(t)PBG(u(t)) + G^T(u(t))B^T Pu(t) \\ & \le +u^T(t)PBRB^T Pu(t) + G^T(u(t))R^{-1}G(u(t)). \end{aligned} \tag{10}$$

Substituting (10) into (9), we have

$$\begin{aligned} & \dot V(u(t), G(u(\gamma(t))), G(u(t-\tau))) \le u^T(t)\Omega u(t) \\ & + G^T(u(\gamma(t)))C^T Pu(t) + u^T(t)PCG(u(\gamma(t))) \\ & + G^T(u(t-\tau))D^T Pu(t) + u^T(t)PDG(u(t-\tau)) \\ & + G^T(u(\gamma(t)))SG(u(\gamma(t))) + G^T(u(t-\tau))QG(u(t-\tau)). \end{aligned}$$

Then we obtain

$$\dot V(u_t) \le -\eta(t)\Sigma\eta^T(t), \tag{11}$$

where $\eta(t) = \left[\, u^T(t) \quad G^T(u(\gamma(t))) \quad G^T(u(t-\tau)) \,\right]$ and

$$\Sigma = \begin{pmatrix} \Omega & -PC & -PD \\ -C^T P & S & 0 \\ -D^T P & 0 & Q \end{pmatrix}.$$

Now we will prove global exponential stability of the system (7). Note that $\ell = \max_{1\le i\le n}\{L_i\}$ for $i = 1, \ldots, n$ and

$$\begin{aligned} V(u_t) \le & \lambda_{\max}(P)||u(t)||^2 + \lambda_{\max}(Q)\ell^2 \int_{t-\tau}^{t} ||u(\xi)||^2 \mathrm{d}\xi \\ & + \lambda_{\max}(S)\ell^2 \int_{\gamma(t)}^{t} ||u(\xi)||^2 \mathrm{d}\xi. \end{aligned} \tag{12}$$

From (8) and (11), one can see that there exists a scalar $m > 0$ such that

$$\begin{pmatrix} \Omega - mI & -PC & -PD \\ -C^T P & S & 0 \\ -D^T P & 0 & Q \end{pmatrix} > 0.$$

Then, we can obtain easily the following equation for any scalar $c > 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(e^{ct}V(u_t)) = & e^{ct}[c(V(u_t)) + \dot V(u_t)] \\ \le & e^{ct}\Big[(c\lambda_{\max}(P) - m)||u(t)||^2 \\ & + c\lambda_{\max}(Q)\ell^2 \int_{t-\tau}^{t} ||u(\xi)||^2 \mathrm{d}\xi \\ & + \lambda_{\max}(S)\ell^2 \int_{\gamma(t)}^{t} ||u(\xi)||^2 \mathrm{d}\xi\Big]. \end{aligned}$$

$\gamma(t)$ is a function of the alternate constancy. If $\theta_k \le t < \zeta_k$, then $\gamma(t) > t$. Similarly, if $\zeta_k \le t < \theta_{k+1}$, then $\gamma(t) < t$. One can obtain $\gamma(t) > t - \tau$, for all $t \in \mathbb{R}$ by applying (A4). Then one can find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(e^{ct}V(u_t)) \le e^{ct}\Big[& (c\lambda_{\max}(P) - m)||u(t)||^2 \\ & + c\vartheta\ell^2 \int_{t-\tau}^{t} ||u(\xi)||^2 \mathrm{d}\xi\Big]. \end{aligned} \tag{13}$$

where $\vartheta = \lambda_{\max}(Q) + \lambda_{\max}(S)$. By integrating two sides from 0 to $T > 0$, we obtain

$$\begin{aligned} e^{cT}V(u_T) - V(u_0) \le & (c\lambda_{\max}(P) - m) \int_0^T e^{ct}||u(t)||^2 \mathrm{d}t \\ & + c\vartheta \int_0^T \int_{t-\tau}^{t} e^{ct}||u(\xi)||^2 \mathrm{d}\xi \mathrm{d}t. \end{aligned} \tag{14}$$

One can see easily that

$$\int_0^T \int_{t-\tau}^t e^{ct}||u(\xi)||^2 \mathrm{d}\xi \mathrm{d}t \le \tau \int_{-\tau}^T e^{c(t+\tau)}||u(t)||^2 \mathrm{d}t$$
$$\le \tau e^{c\tau} \int_{-\tau}^0 ||u(t)||^2 \mathrm{d}t + \tau e^{c\tau} \int_0^T e^{ct}||u(t)||^2 \mathrm{d}t.$$

Then we obtain

$$e^{cT}V(u_T) \le (c\lambda_{\max}(P) - m + c\vartheta\ell^2\tau e^{c\tau}) \int_0^T e^{ct}||u(t)||^2 \mathrm{d}t + c\vartheta\ell^2\tau e^{c\tau} \int_{-\tau}^0 ||u(t)||^2 \mathrm{d}t + V(u_0).$$

By choosing a scalar $c > 0$ such as $m = c\lambda_{\max}(P) + c\vartheta\ell^2\tau e^{c\tau}$, we have

$$e^{cT}V(u_T) \le c\vartheta\ell^2\tau e^{c\tau} \int_{-\tau}^0 ||u(t)||^2 \mathrm{d}t + V(u_0). \tag{15}$$

$V(u_0)$ satisfies the following inequality from the definition of $V(u_t)$

$$V(u_0) \le \lambda_{\max}(P)||u_0||^2 + \vartheta\ell^2 \int_{-\tau}^0 ||u(\xi)||^2 \mathrm{d}\xi. \tag{16}$$

Substituting (16) into (15), we have

$$e^{cT}V(u_T) \le (c\vartheta\ell^2\tau^2 e^{c\tau} + \lambda_{\max}(P) + \vartheta\ell^2\tau) \sup_{-\tau \le \xi \le 0} ||u(\xi)||^2.$$

Also we know from (12), $\lambda_{\min}(P)||u(T)||^2 \le V(u_T)$. Consequently, we have

$$||u_T)|| \le \kappa e^{-cT/2} \sup_{-\tau \le \xi \le 0} ||u(\xi)||^2.$$

where $\kappa = \sqrt{\frac{c\vartheta\ell^2\tau^2 e^{c\tau} + \lambda_{\max}(P) + \vartheta\ell^2\tau}{\lambda_{\min}(P)}}$.

The theorem is proved. □

3 Examples and numerical simulations

In this section, we give two examples.

Example 1 Consider the following model

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.01 & 0.02 \\ 0.03 & 0.01 \end{pmatrix}\begin{pmatrix} \tan h(x_1(t)) \\ \tan h(x_2(t)) \end{pmatrix} + \begin{pmatrix} 0.01 & 0.02 \\ 0.02 & 0.03 \end{pmatrix}\begin{pmatrix} \tan h(x_1(\gamma(t))) \\ \tan h(x_2(\gamma(t))) \end{pmatrix} + \begin{pmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}\begin{pmatrix} \tan h(x_1(t-2)) \\ \tan h(x_2(t-2)) \end{pmatrix}. \tag{17}$$

If $L_1 = 0.1$ and $L_2 = 0.1$, it can be shown easily that (17) satisfies the conditions for the existence of a unique equilibrium of (17). For

$$P = \begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix},\ Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$
$$R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},\ S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

the condition of the Theorem 1 is satisfied. So, the unique equilibrium $x^* = [0.0064, 0.0084]^T$ of the system (17) is globally exponentially stable (Fig. 2). Also the phase portrait in Fig. 3 demonstrates the existence of the equilibrium point of the system (17). In Fig. 3, one can see by the simulation that the trajectory with initial point $[2.8, 2]^T$ approaches to the equilibrium $[0.0064, 0.0084]^T$ as time increases.

Example 2 Consider the following model

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix} 20 & 0 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \tan h(x_1(t)) \\ \tan h(x_2(t)) \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \tan h(x_1(\gamma(t))) \\ \tan h(x_2(\gamma(t))) \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \tan h(x_1(t-2)) \\ \tan h(x_2(t-2)) \end{pmatrix}. \tag{18}$$

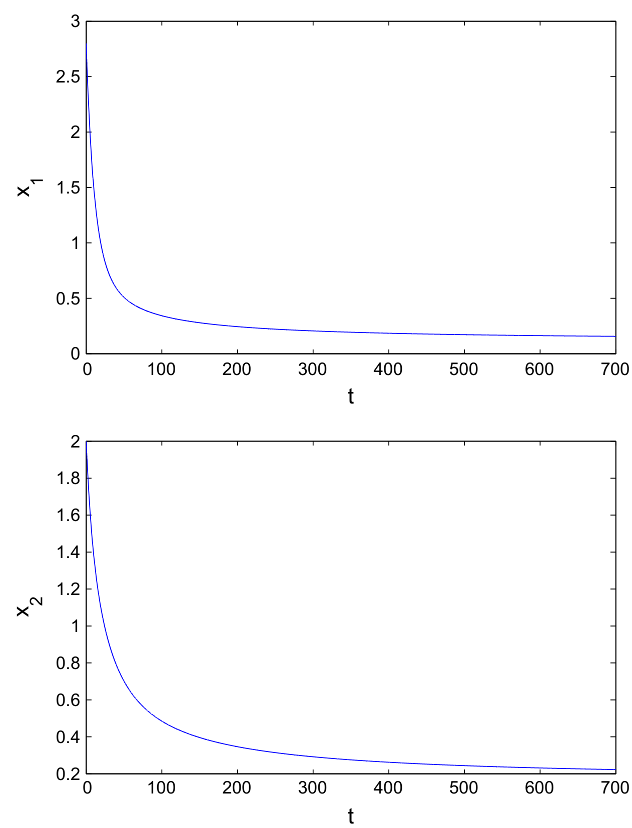

Fig. 2 Time response of $x_1(t)$ and $x_2(t)$ with piecewise constant argument in Example 1

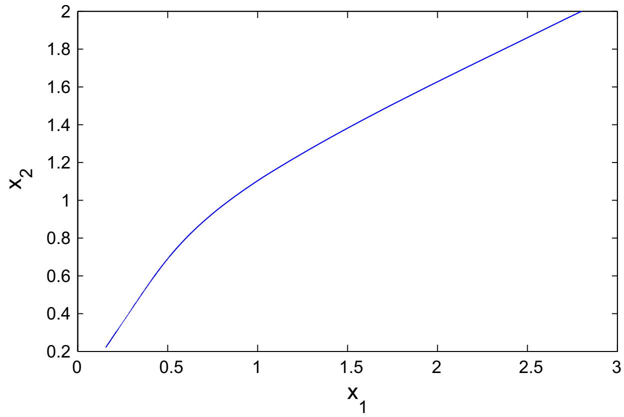

Fig. 3 The phase portrait of the system in Example 1

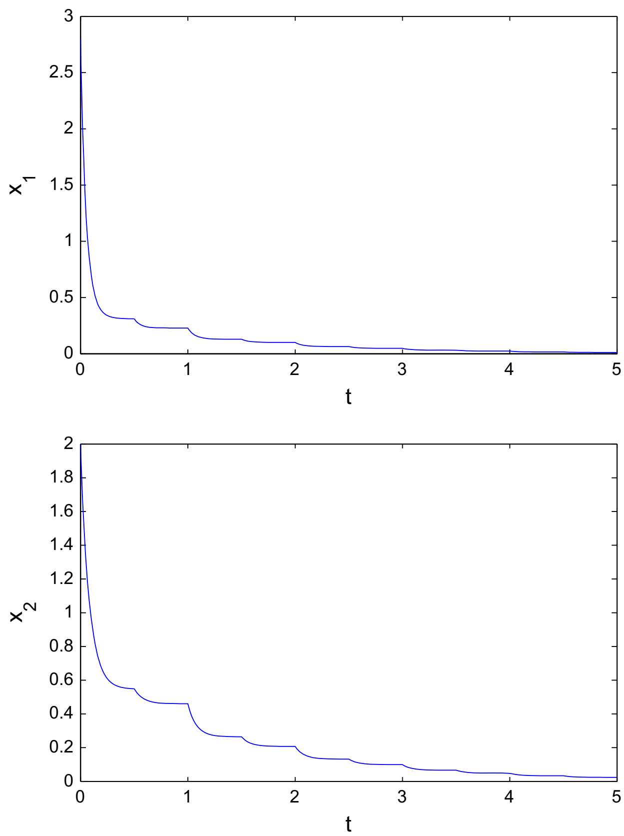

Fig. 4 Time response of $x_1(t)$ and $x_2(t)$ with piecewise constant argument in Example 2

Here we choose the Lipschitz constants such that $L_1 = 1$ and $L_2 = 1$. The parameters are chosen large to show the influence of the piecewise constant argument. In this example, we want to illustrate the non-smoothness, although the condition of Theorem 1 is not satisfied with these coefficients and Lipschitz constants. Fig. 4 makes clear the non-smoothness of the solution with the initial point $[2.8, 2]^T$ at the switching points θ_k; $k \in N$. We conclude that the small parameters prevent to see the non-smoothness, precisely. One can see that the solution converges to the unique equilibrium such that $x^* = [0.0115, 0.0239]^T$.

4 Conclusion

In this paper, a new Hopfield neural network with piecewise constant argument of generalized type and constant delay has been studied. Up to now, various kinds of delays were introduced to the Hopfield neural network systems such that constant delays, single time delays, time-varying delays and distributed delays. But it is the first time that a Hopfield neural network with both piecewise constant argument of generalized type and constant delay is considered. The crucial point about this paper is that an LMI method has been extended to a multi-compartmental structure to investigate the stability of the system. Two examples are given to illustrate our results.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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