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ABSTRACT

Dynamics are constructed for fractals utilizing the motion associated with Duffing equation. Using the paradigm of Fatou-Julia iteration, we develop iterations to map fractals accompanied with a criterion to ensure that the image is again a fractal. Because of the close link between mappings, differential equations and dynamical systems, one can introduce dynamics for fractals through differential equations such that they become points of the solution trajectory. There is no doubt that the differential equations have a distinct role for studying chaos. Therefore, characterization of fractals as trajectory points is an important step toward a better understanding of the link between chaos and fractal geometry. Moreover, it would be helpful to enhance and widen the scope of their applications in physics and engineering.

Fractal and chaos are interesting features that characterize many natural phenomena, and they have a significant impact on practical applications. Even though many studies have investigated the relationship between chaos and fractals, still there is no formal theory to interpret this link. The link is more clear when fractal dimension is used to measure the extent to which a trajectory fills its phase space.[1] In this research, we propose to connect the dynamics of differential equations with fractals. For this purpose, a fractal mapping iteration is developed such that the motion associated with Duffing equation is considered as a map with a fractal as an initial set. Thus, the orbit corresponding to the solution of the differential equation can be seen as a continuous sequence of fractals.

I. INTRODUCTION AND PRELIMINARIES

The term "fractal" was coined by Benoit Mandelbrot in 1975.[2] He defined fractal as a set for which the Hausdorff dimension strictly exceeds the topological dimension. Dealing with fractals goes back to the 17th century when Gottfried Leibniz introduced the notions of recursive self-similarity.[3] A considerable leap in the construction of fractals was performed in 1883 by Georg Cantor, as he discovered the most essential and influential fractal known as the Cantor set.

Waclaw Sierpinski was one of the mathematicians who made significant contributions in the field of fractals. He introduced the famous square fractal in 1916,[4] known as the Sierpinski carpet. The fractal is generated by a recursive process of removing symmetrical parts from an initial square. The process starts with subdividing a solid square into nine identical subsquares and then removing the central one. In the next iterations, the same procedure is repeated to each of the remaining squares from the preceding iteration. In an analogous way to the carpet, Sierpinski developed a triangular fractal known as the Sierpinski gasket.

Involvement of the dynamics of iterative maps in fractal construction was a critical step made by the French mathematicians Pierre Fatou and Gaston Julia around 1917–1918, during their independent studies on the iteration of rational functions in the complex plane.[5,6] They described what we call today the Fatou-Julia iteration (FJI).[7] The iteration is defined over a domain $\mathcal{D} \subseteq \mathbb{C}$ by

$$z_{n+1} = F(z_n), \qquad (1)$$

where $F : \mathcal{D} \to \mathcal{D}$ is a given function for the construction of the fractal set $\mathcal{F}$. The points $z_0 \in \mathcal{D}$ are included in the set $\mathcal{F}$ depending on the boundedness of the sequence z_n, $n = 0, 1, 2, \ldots$, and we say that the set $\mathcal{F}$ is constructed by FJI.

In practice, one cannot verify the boundedness for infinitely long iterations. This is why in simulation we fix an integer k and a bounded subset $M \subset \mathbb{C}$, then the obtained set is the collection of all points $z_0 \in \mathcal{D}$ such that the points z_n, $n = 1, 2, \ldots, k$, belong to M. In what follows, we call such a set a *kth* "approximation" of the fractal $\mathcal{F}$.

The most popular fractals, Julia and Mandelbrot sets, are generated using the iteration of the quadratic map $F(z_n) = z_n^2 + c$, where c

is a complex number. Julia sets contain the points $z_0 \in \mathbb{C}$ corresponding to the bounded sequence z_n, whereas the Mandelbrot set is the set of parameter values $c \in \mathbb{C}$ such that $\{z_n\}$, $z_0 = 0$, remains bounded.

To map fractals, a new method attained by FJI is suggested. The method is based on involving a map Φ in the iteration (1) to define the Fractal Mapping Iteration (FMI)

$$\Phi^{-1}(\mathbf{x}_{n+1}) = F\big(\Phi^{-1}(\mathbf{x}_n)\big), \tag{2}$$

where $\mathbf{x}_0$ is a number in $\mathbb{C}$ or in $\mathbb{R}^2$. In this recursive equation, the FJI is applied to the preimage of the mapped set. To determine a kth approximation of the mapped fractal $\mathcal{F}_\Phi$, we consider a bounded subset $M \subset \mathcal{D}$, where $\mathcal{D}$ is the domain of the original FJI. The boundedness of the sequence $\Phi^{-1}(\mathbf{x}_n)$, $n = 1, 2, \ldots, k$, in M, is examined for all points $\mathbf{x}_0 \in \mathcal{D}$. Then, the kth approximation of $\mathcal{F}_\Phi$ is the set containing only the points $\mathbf{x}_0$ corresponding to the bounded sequences. The FMI (2) is described in a general form, i.e., it is valid for any function F.

In the present paper, we shall apply the FMI to fractals and construct continuous dynamics using differential equations, namely, the classical Duffing equation. As examples, the Julia set and the Sierpinski carpet are considered. To map the Julia set, the FMI takes the form

$$\Phi^{-1}(z_{n+1}) = \big(\Phi^{-1}(z_n)\big)^2 + c. \tag{3}$$

The mapped Julia set $\mathcal{F}_\Phi$ is the set of parameter values $z_0 \in \mathbb{C}$ such that the sequence $\Phi^{-1}(z_n)$ remains bounded.

For the Sierpinski carpet, the idea of FJI is adopted to develop a scheme for constructing the set. The technique of the FJI is based on detecting the points of a fractal set through the boundedness of their iterations under a specific map. Here, we shall extend the technique to include any possible criterion for grouping points in a given domain. We use a map that constructs a set that is similar to the Cantor set in the generation way but different in structure. The purpose of such sets is to cut out successively smaller parts (holes) in the Sierpinski carpet. This is why we call these types of sets as "perforation sets."

Let us introduce the map

$$\psi_n(x) = B\sin(A_n x), \tag{4}$$

where $A_n = \pi a^{n-1}$, $B = \frac{\pi}{b}$, and a, b are parameters. The recursive formula is defined as follows:

$$\psi_0(x_0) := x_0,$$
$$x_n = \psi_n(x_0), \quad n = 1, 2, \ldots .$$

To construct the perforation set, we start with the interval $\mathcal{I} = [0, 1]$ and include in the kth approximation of the set each point $x_0 \in \mathcal{I}$ that satisfies $|x_k| \le 1$. Thus, for the Sierpinski carpet, we use a two-dimensional version of the map (4) which can be defined in the form

$$\psi_n(x, y) = \big(B\sin(A_n x),\ B\sin(A_n y)\big). \tag{5}$$

The procedure here is to determine the image sequence (x_n, y_n) of each point $(x_0, y_0) \in \mathcal{D}$, i.e., $(x_n, y_n) = \psi_n(x_0, y_0)$. If we choose $\mathcal{D} = [0, 1] \times [0, 1]$, the point (x_0, y_0) is excluded from the set if the condition, $|x_n| > 1$, $|y_n| > 1$, is satisfied for some $n \in \mathbb{N}$. For the values of the parameters $a = 3$ and $b = 3$, the scheme gives the classical Sierpinski carpet.

Now, let $\Phi : \mathcal{D} \to \mathcal{D}'$ be an invertible function. Then, the fractal mapping scheme can be defined by

$$\Phi^{-1}(\xi_n, \eta_n) = \psi_n\big(\Phi^{-1}(\xi_0, \eta_0)\big). \tag{6}$$

In order to obtain the mapped Sierpinski carpet, $\mathcal{F}_\Phi$, the domain of (6) is restricted only to the points (ξ_0, η_0) that belong to the mapped domain $\mathcal{D}'$. Thus, if we let $(u_n, v_n) = \Phi^{-1}(\xi_n, \eta_n)$, then (ξ_0, η_0) is included in $\mathcal{F}_\Phi$ if at least one of $|u_n|$ and $|v_n|$ is less than or equal to 1 for all $n \in \mathbb{N}$.

One can show that, for the FMIs (3) and (6), the set $\mathcal{F}_\Phi$ is merely the image of $\mathcal{F}$ under the map Φ. However, the following question arises here: Is the mapped set a fractal? The answer is "yes" if the map Φ satisfies a bi-Lipschitz condition. This result is stated in the following lemma.

Lemma 1. [8] *Let $E \subseteq \mathbb{R}^n$. If $f : E \to \mathbb{R}^m$ is a bi-Lipschitz function, i.e., there exist real numbers $l_1, l_2 > 0$ such that $l_1|u - v| \le |f(u) - f(v)| \le l_2|u - v|$, for all $u, v \in E$, then*

$$\dim_H\big(f(E)\big) = \dim_H(E),$$

where $\dim_H$ denotes the Hausdorff dimension.

II. MAIN RESULT

We connect differential equations with fractals by involving their dynamics as maps in FMI. The general idea of constructing a dynamics for fractal is to use the motion of a dynamical system with a fractal as an initial set. The motion of dynamical system is defined by $\mathcal{A}_t\mathbf{x}_0 = \varphi(t, \mathbf{x}_0)$, where φ is the solution of a two-dimensional system of ordinary differential equations,

$$\mathbf{x}' = g(t, \mathbf{x}), \tag{7}$$

with $\varphi(0, \mathbf{x}_0) = \mathbf{x}_0$. Thus, we construct dynamics of sets $\mathcal{A}_t\mathcal{F}$, where the fractal $\mathcal{F}$ is the initial value. Through this procedure, the differential equations are involved in fractals such that the latter become points of the solution orbits. If the map $\mathcal{A}_t$ is bi-Lipschitzian [this is true, for instance, if the function g in (7) is Lipschitzian] then the set $\mathcal{A}_t\mathcal{F}$ for each fixed t is a fractal.

As our differential equation, we shall consider the Duffing equation

$$u'' + \delta u' + \beta u + \alpha u^3 = \gamma\cos\omega t, \tag{8}$$

where $\delta, \beta, \alpha, \gamma$, and ω are real parameters. Using the variables $x = u$ and $y = u'$, one can show that Eq. (8) is equivalent to the nonautonomous system

$$\begin{aligned} x' &= y, \\ y' &= -\delta y - \beta x - \alpha x^3 + \gamma\cos\omega t. \end{aligned} \tag{9}$$

Let us denote by $\big(x(t, x_0), y(t, y_0)\big)$ the solution of (9) with $x(0, x_0) = x_0$, $y(0, y_0) = y_0$. System (9) can be numerically solved to construct a dynamical system with the motion $\mathcal{A}_t(x_0, y_0) = (A_t x_0, B_t y_0)$, where $A_t x_0 = x(t, x_0)$ and $B_t y_0 = y(t, y_0)$.

Applying this dynamics for the Julia set, the FMI (3) becomes

$$z_{n+1} = \mathcal{A}_t\Big(\big(\mathcal{A}_{-t}(z_n)\big)^2 + c\Big),$$

where if we let $z = x + iy$, then $\mathcal{A}_t(z) = A_t x + iB_t y$. For numerical simulation, we consider an approximation of the Julia set, with

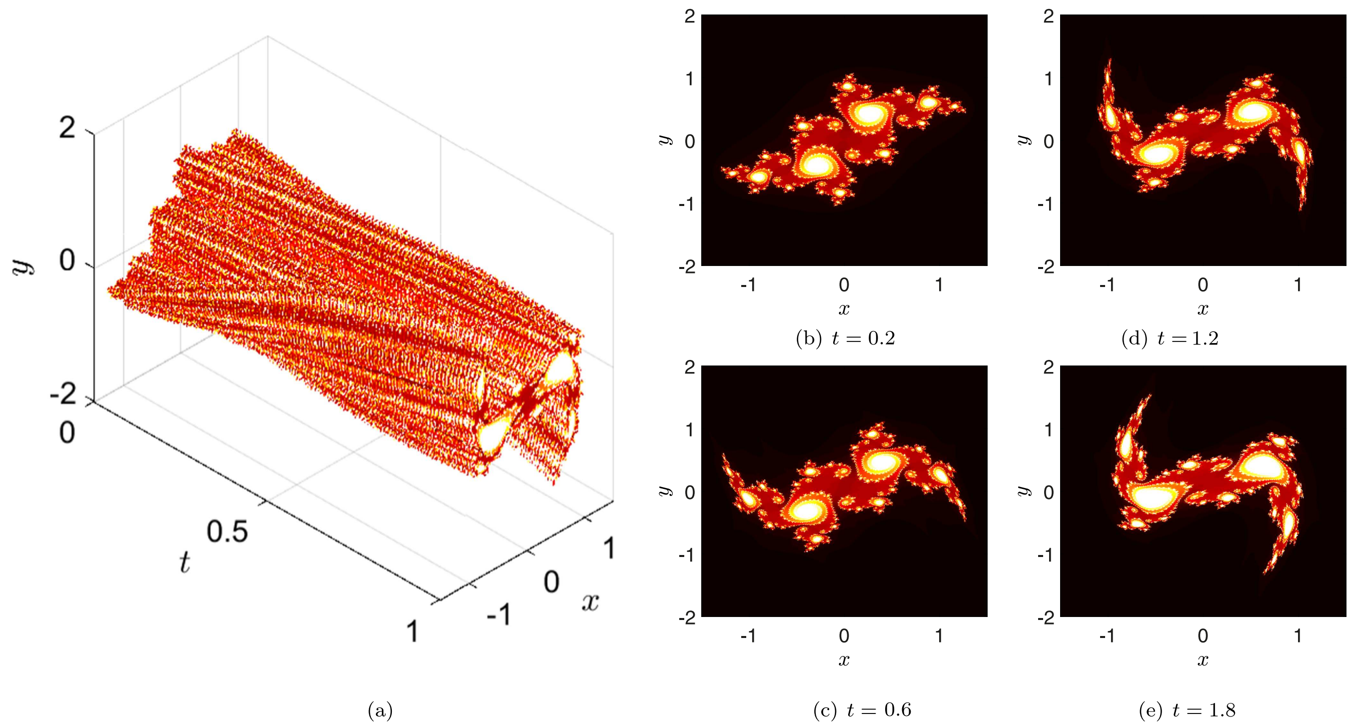

FIG. 1. The trajectory of the Duffing dynamics for the Julia set and its sections.

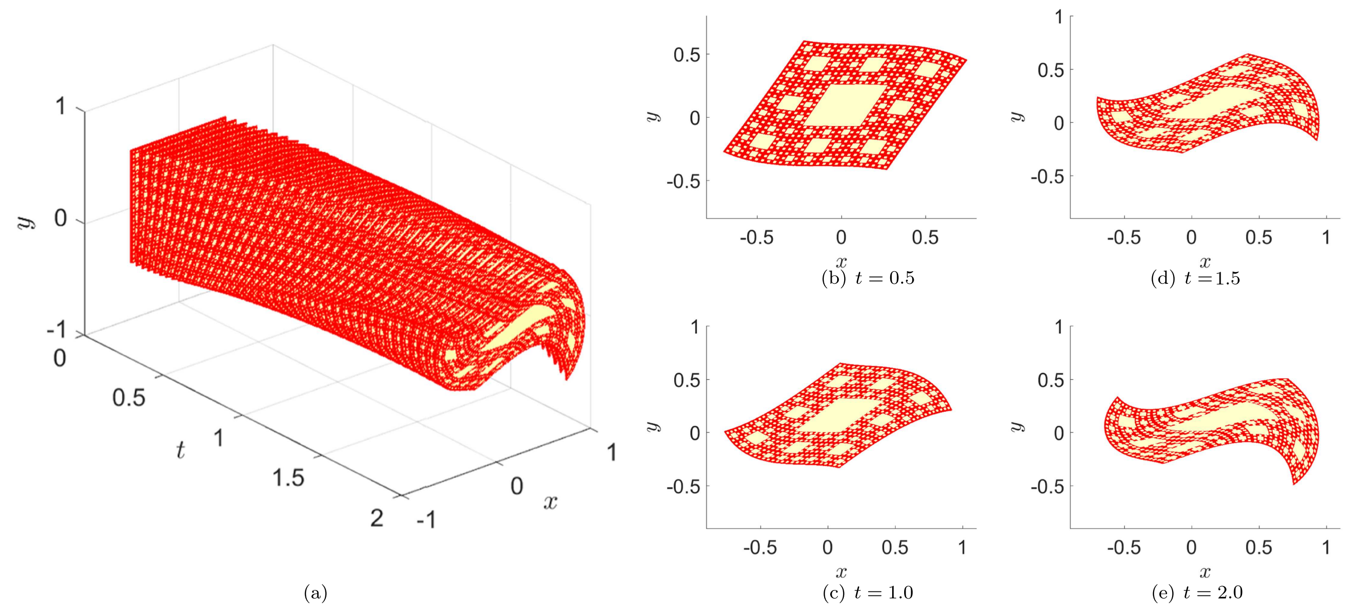

FIG. 2. The trajectory of the Duffing dynamics for the Sierpinski carpet and its sections.

$c = -0.175 - 0.655i$, as an initial set, and use the parameter values $\delta = 0.04$, $\beta = 0$, $\alpha = 1.5$, $\gamma = 0.2$, and $\omega = 1$ in system (9). The fractal's trajectory for $0 \leq t \leq 1$ is shown in Fig. 1(a). Figures 1(b)–1(e) exhibit the sections of the trajectory at the moments $t = 0.2, t = 0.6, t = 1.2$, and $t = 1.8$.

In the case of the Sierpinski carpet, we iteratively apply a motion $\mathcal{A}_t$ to the scheme (5) to obtain the FMI

$$(\xi_n, \eta_n) = \mathcal{A}_t\Big(\psi_n\big(\mathcal{A}_{-t}(\xi, \eta)\big)\Big),$$

where $\mathcal{A}_t(x, y) = (A_t x, B_t y)$. The iteration is applied for an approximation of the Sierpinski carpet as an initial set. The fractal trajectory for $0 \leq t \leq 2$ is shown in Fig. 2(a), whereas Figs. 2(b)–2(e) display the sections of the trajectory at the specific times $t = 0.5$, $t = 1.0$, $t = 1.5$, and $t = 2.0$. The values $\delta = 0.08$, $\beta = 0$, $\alpha = 1$, $\gamma = 0.2$, and $\omega = 1$ are used in the simulation.

III. CONCLUSION

In this paper, we propose to connect fractals with differential equations by considering dynamics of the Duffing equation as a trajectory initiated at a fractal set. The dynamics are constructed using the fractal mapping iteration, which is defined on the basis of the Fatou-Julia iteration. We consider the Julia set and the Sierpinski carpet as two examples of the fractal sets. For the Sierpinski carpet, an iteration scheme is constructed to generate the set, then a fractal mapping scheme is formulated. Since the Duffing equation possesses chaotic solutions for specific values of the parameters in (8), the results of the research can be useful for investigating the fractal nature of the chaotic attractors.

Important applications can be considered by taking into account the relationship between the fractal theory of motion and quantum mechanics. In the scale relativity theory,[9,10] fractals are considered as a geometric framework of atomic scale motions such that the quantum behavior can be viewed as particles moving on fractal trajectories. One can suppose that by composing the scale relativity theory with dynamics of fractals developed in this paper, we will be able to understand better the fractal nature of the world. A possible connection between fractal mappings and quantum mechanics through the scale relativity theory can provide important applications for the former in various fields such as biology, cosmology, and fractal geodesics (see Ref. 10 and the relevant references therein).

Owing to the important roles of Sierpinski fractals in several applications like weighted networks, trapping problems, antenna engineering, city planning, and urban growth,[11–15] we expect that the results of the present study will be helpful in the fields of applications. One of the crucial applications of fractals involves optimization theory. Fractal geometry is used to solve some classes of optimization problems such as supply chain management and hierarchical design.[16,17] In Ref. 16, for instance, the properties of a particular hierarchical structure are established. The authors constructed the relationship between the Hausdorff dimension of the optimal structure and loading for which the structure is optimized. The Hausdorff dimension is calculated through considering the self-similarity of the structure at different hierarchical levels. The self-similar fractals, like the Sierpinski carpet, are considered as effective tools for studying the hierarchical structures.[13,18] Thus, finding a way to map this type of structures allows to create a new hierarchical structure with the same Hausdorff dimension but different mechanical properties if one considers bi-Lipschitz maps.

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