An Algorithm for Finding Core in Assignment Games

Serkan Küçükşenel*

Abstract

This paper presents an algorithm that finds all core payoffs in the assignment games with money. Our algorithm provides an easy way to reach all core outcomes using the similar fixed point construction arguments from the two-sided matching literature.

JEL Classification Numbers: C78.

Key Words: Two-sided matching, assignment game, core, algorithm.

*Department of Economics, Middle East Technical University, Ankara, 06531, Turkey. E-mail: kuserkan@metu.edu.tr; Tel: +90 312 210 3013; Fax: +90 312 210 7964.

1 Introduction

An assignment game is a special case of two-sided matching markets in which monetary transfers are allowed. In this market, agents on one side of the market are matched with the other agents on the other side of the market. We label without loss of generality these two sides as firms and workers. The matching is one to one and monetary transfers (wage payments) are allowed. Since the only allowed matching type is one to one, a worker can only be matched (or work) to one firm, and each firm can employ only one worker. One commonly used solution concept for such markets is the core. The core outcomes specify which bilateral employment agreements we can expect to observe, and how the agents divide their gains. In this paper, we construct the first algorithm that reaches to the all core outcomes for assignment games with money.

Shapley and Shubik (1972) show that every assignment game has non-empty core and core payoffs have a nice structure. The payoff structure is a non-empty complete lattice, and there is a polarization of interests in the core. This means that there is a stable outcome which is the most preferred by every agent on one side of the market and at the same time it is the least preferred by every agent on the other side of the market. Geometrically, the core is a closed, convex polyhedron whose dimension is equal to at most the minimum of the number of members in one group or in the other (Shapley and Shubik (1972)).

Kucuksenel (2011) constructs a map T on the set of feasible payoffs such that the set of fixed points of T is equal to the core outcomes. In this paper, we construct the first algorithm to find all core outcomes by iterating T for the assignment game. This type of fixed point argument has been used in assignment problems with side payments before, but they only characterized certain points in the interior of the core (a subset of the core—*symmetrically bargained allocations*) as stationary points of a rebargaining process between players (Rochford (1984)). Moreover, fixed point methods have been used in matching markets without side transfers (NTU games), see for example Adachi (2000), Echenique and Oviedo (2004), Echenique and Oviedo (2006) or Echenique and Yenmez (2007) for applications of a fixed point approach for different environments. The algorithms to find core outcomes are also provided in mentioned studies related to assignment problems without side transfers. Different algorithms to find only extreme points of core outcomes in two-sided matching problems with one-way monetary transfers are also provided (Afacan (2013), Abizada (2016)).

The organization of the rest of the paper is as follows: In the next section, we give a brief introduction to the Shapley and Shubik assignment game and provide some of the well-known results using linear programming formulation. In Section 3, we present the formulation in Kucuksenel (2011) to represent the core as fixed points of a map. In Section 4, we introduce the algorithm and show that the algorithm reaches to the all possible core outcomes in the assignment games. Section 5 shows that the extension of the formulation using core outcomes is not possible. The discussion and future research agenda follows in Section 6.

2 Assignment games with money

This section gives a brief description of the assignment games and provides some well-known results via linear programming proofs. We refer the reader to Shapley and Shubik (1972) or Roth and Sotomayor (1990) for more discussion and justification of the setup. Our exposition mainly follows Kucuksenel (2011) in this section.

The game in coalitional function form with side payments is defined by three-tuple $\Gamma = \langle F, W, \alpha \rangle$ where
1. $F = \{f_1, ..., f_m\}$ is a set of firms,
2. $W = \{w_1, ..., w_n\}$ is a set of workers,
3. α is a $m \times n$ matrix of nonnegative numbers $\{\alpha_{fw} \in \mathbb{R}_+ : (f, w) \in F \times W\}$ where α_{fw} is the value of pairwise partnership. Note that $\alpha_{kk} = 0$ for all $k \in F \cup W$.

An **assignment** $\mu : F \cup W \to F \cup W$ is a one-to-one mapping of order two (that is $\mu^2(k) = k$) such that if $\mu(f) \neq f$ then $\mu(f) \in W$ and if $\mu(w) \neq w$ then $\mu(w) \in F$. Let $\mathcal{M}$ be the set of all assignments. An assignment μ can also be represented as a vector $x \in \{0, 1\}^{F \times W}$, such that $x_{fw} = 1$ if $\mu(f) = w$ and $x_{fw} = 0$, otherwise. Hence, $\sum_{w \in W} x_{fw} \leq 1$ for all $f \in F$ and $\sum_{f \in F} x_{fw} \leq 1$ for all $w \in W$.

An assignment x is **optimal** if for all $x' \in \mathcal{M}$, $\sum_{(f,w) \in F \times W} \alpha_{fw} x_{fw} \geq \sum_{(f,w) \in F \times W} \alpha_{fw} x'_{fw}$. Let $\mathcal{X}$ be the set of optimal assignments. The optimal assignment is usually unique. If there is more than one optimal assignment, a slight perturbation of the values of the pairwise partnerships will result in a unique optimal assignment.

Any agent is free to remain single and receive zero, and the worth of an arbitrary coalition equals to the sums of the pairwise coalitions it can form with pairs consisting of one agent from F and one from W. That is for all coalitions S,

$$V(S) = \begin{cases} 0 & \text{if } |S| = 0 \text{ or } 1 \\ 0 & \text{if } S \subseteq F \text{ or } S \subseteq W \\ \max_{\mu: F\cap S \to W\cap S} \sum_{f \in F\cap S} \alpha_{f\mu(f)} & \text{if } |F \cap S| \leq |W \cap S| \\ \max_{\mu': W\cap S \to F\cap S} \sum_{w \in W\cap S} \alpha_{\mu(w)w} & \text{if } |F \cap S| \geq |W \cap S|. \end{cases}$$

Definition 1 *The pair of vectors (u, v), with $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, is a feasible payoff for $\Gamma = \langle F, W, \alpha \rangle$ if there is an assignment x such that*

$$\sum_{f \in F} u_f + \sum_{w \in W} v_w = \sum_{(f,w) \in F \times W} \alpha_{fw} x_{fw}.$$

In this case we say (u, v) and x are **compatible** with each other, and we call $((u, v); x)$ a feasible outcome.

Definition 2 *A feasible outcome $((u, v); x)$ is **stable** (or the payoff (u, v) with an assignment x is stable) if*
(i) $u_f \geq 0, \quad v_w \geq 0$ *(individual rationality)*
(ii) $u_f + v_w \geq \alpha_{fw}$ *for all* $(f, w) \in F \times W$.

Note that condition (ii) only eliminates deviations by pair of agents since the set of pairwise stable outcomes coincides with the set of group stable outcomes in this framework. Let $S(\Gamma)$ be the set of stable payoffs.

Consider just the assignment problem for the coalition of all players:

(AP)

$$\begin{aligned} \max \qquad & z = \sum_{(f,w) \in F \times W} \alpha_{fw} x_{fw} \\ s.t. \qquad & \sum_{w \in W} x_{fw} \leq 1 \ \ \forall \ \ f \in F, \\ & \sum_{f \in F} x_{fw} \leq 1 \ \ \forall \ \ w \in W, \\ & x_{fw} \geq 0 \ \ \forall \ \ (f, w) \in F \times W. \end{aligned}$$

This optimization problem is associated with dual linear program having the form:

(DAP)

$$\begin{aligned} \min \quad & d = \sum_{f \in F} u_f + \sum_{w \in W} v_w \\ s.t. \quad & u_f + v_w \geq \alpha_{fw} \ \ \forall \ \ (f, w) \in F \times W, \\ & u_f, v_w \geq 0. \end{aligned}$$

Therefore, (DAP) formulates the problem of finding payoff vectors in the core of the assignment game. The existence of optimal solutions of (AP) and duality theorem show that the set of stable payoff vectors is nonempty. Moreover, in the game the set of stable outcomes and the core are the same.

Theorem 1 (Shapley and Shubik (1972)) *The core of an assignment game is nonempty and is precisely equal to the set of solutions of the (DAP).*

3 The T Mapping

In this section, we present the formulation in Kucuksenel (2011) that fully characterize the core as the set of fixed points of a certain function. We assume that $|F| = |W| = n$ to simplify the formulation. We shall also assume that for all $i \in F$ $u_i \in \{0, 1, ..., \max_{j \in W} \alpha_{ij}\}$ and for all $j \in W$ $v_j \in \{0, 1, ..., \max_{i \in F} \alpha_{ij}\}$ to make the payoff space discrete. These assumptions simplify the notation, but all results hold without these assumptions.

We can now proceed to define the formulation by following the identical mathematical notation in Kucuksenel (2011). A firm-permutation is a bijection π_F from F to F, and a worker-permutation is a bijection π_W from W to W. This type of permutations is useful in defining the order of agents. We denote $\pi_f^{-1}(i) \in F$ as the i-th firm and $\pi_w^{-1}(j) \in W$ as the j-th worker. Let Y be the set of possible payoffs such that:

$$Y = \{((u_i)_{i \in F}, (v_j)_{j \in W}) \mid \forall i \in F, \ \ 0 \leq u_i \leq \max_{j \in W} \alpha_{ij} \ ; \ \forall j \in W, \ \ 0 \leq v_j \leq \max_{i \in F} \alpha_{ij}\}.$$

Given $(u, v), \pi_F$, and π_W, let

$$U(u_{\pi_F^{-1}(i)}, v) = \max_{j \in W^i} (\alpha_{\pi_F^{-1}(i) j} - v_j),$$

$$V(u, v_{\pi_W^{-1}(j)}) = \max_{i \in F^j}(\alpha_{i\pi_W^{-1}(j)} - u_i),$$

where $W^1 = W$ and for all $i \geq 2$

$$W^i = W^{i-1} \setminus min\{j : j \in argmax(\alpha_{\pi_F^{-1}(i-1)j} - v_j)\},$$

and $F^1 = F$ and for all $j \geq 2$

$$F^j = F^{j-1} \setminus min\{i : i \in argmax(\alpha_{i\pi_W^{-1}(j-1)} - u_i)\}.$$

It is possible that the outcome of the mapping depends on the order of players. For some $(u, v) \in Y$, the outcome of the mapping does not depend on the exact order of players. Let $(u, v) \in \mathcal{F}$ if there exists a tie breaking rule, π_F, such that for all $i \in F$, $\pi_F(i, argmax_{j \in W}(\alpha_{ij} - v_j)) \in W^i$. We call π_F a firm-consistent tie breaking rule. If there is a firm-consistent tie breaking rule, $U(.)$ is independent of the order of firms. That is, $U(u_i, v) = \max_{j \in W^i}(\alpha_{ij} - v_j) = \max_{j \in W}(\alpha_{ij} - v_j)$. Let $(u, v) \in \mathcal{W}$ if there exists a worker-consistent tie breaking rule, π_W, such that for all $j \in W$, $\pi_W(j, argmax_{i \in F}(\alpha_{ij} - u_i)) \in F^j$. This implies $V(.)$ is independent of the order of workers. Thus, $V(u, v_j) = \max_{i \in F^j}(\alpha_{ij} - u_i) = max_{i \in F}(\alpha_{ij} - u_i)$. If there exist both firm and worker consistent tie breaking rules, then let $(u, v) \in \mathcal{B}$. Hence, $\mathcal{B} = \mathcal{F} \cap \mathcal{W}$. Let also $(u, v) \in \mathcal{Z}$ if and only if there does not exist any types of consistent tie breaking rules. We call $\mathcal{F}$ as the set of firm-order independent payoffs, $\mathcal{W}$ as the set of worker-order independent payoffs, $\mathcal{B}$ as the set of order independent payoffs, and $\mathcal{Z}$ as the set of order dependent payoffs. Note that $\mathcal{Z} = Y \setminus (\mathcal{F} \cup \mathcal{W})$. If $(u, v) \notin \mathcal{F}$ ($(u, v) \notin \mathcal{W}$), we use a fixed tie breaking rule $\pi_F(i, S) = \min\{j : j \in S\}$ for all $S \subseteq W$ ($\pi_W(j, S) = \min\{i : i \in S\}$ for all $S \subseteq F$). We refer the reader to Kucuksenel (2011) for more details and examples related to the tie breaking rules.

Let $T : Y \rightarrow Y$ be a mapping such that $T_i(u, v) = U(u_i, v) \vee 0$ if $i \in F$ and $T_j(u, v) = U(u, v_j) \vee 0$ if $j \in W$. Moreover, denote $\mathcal{E}(T) = \{(u, v) \in Y : (u, v) = T(u, v)\}$ as the set of fixed points of T, and $\mathcal{E}^A(T) = \{(u, v) \in A \subseteq Y : (u, v) = T(u, v)\}$ be the set of fixed points of T in the set of payoffs $A \in \{\mathcal{F}, \mathcal{W}, \mathcal{B}, \mathcal{Z}\}$. Given a payoff vector (u, v), T first searches for consistent tie breaking rules. If it is not possible to find a consistent tie breaking then T uses the tie breaking rule defined for the set of order dependent payoffs.

The main result about the mapping can now be stated. The following result shows that the core (or stable) payoffs of the assignment game are equal to the set of fixed points of the aforementioned mapping. Note that core outcomes are the Cartesian product of the core payoffs and the set of optimal assignments.

Proposition 1 (Kucuksenel (2011)) $\mathcal{E}^{\mathcal{B}}(T) = S(\Gamma) = C(\Gamma)$.

We know define the following binary relation $\succeq_F$ on Y.

Definition 3 *Let $(u, v) \in Y$. Define a partial ordering $\succeq_F$ by*

$$(u, v) \succeq_F (u', v') \Leftrightarrow (u, v) \geq_F (u', v') \text{ and } (u', v') \geq_W (u, v).$$

The following lemma about the structure of the core is useful for the next section.

Lemma 1 *Let $(u, v) \in \mathcal{E}^{\mathcal{B}}(T)$, $(u', v') \in \mathcal{E}^{\mathcal{B}}(T)$, and $(u, v) \succeq_F (u', v')$. If $u_f - u'_f = t$, then there is $w \in W$ such that $v'_w - v_w = t$.*

Proof. Since $(u, v) \in \mathcal{B}$ and $(u', v') \in \mathcal{B}$, T is order independent. $T_f(u, v) = \max_{w \in W}(\alpha_{fw} - v_w) \vee 0 = u_f$ and $T_f(u', v') = \max_{w \in W}(\alpha_{fw} - v'_w) \vee 0 = u'_f = u_f - t$. This implies $u_f = \max_{w \in W}(\alpha_{fw} - v'_w + t)$. Therefore, there is $w \in W$ such that $v_w = v'_w - t$. ■

4 The New Algorithm

The T-algorithm uses the formulation of T mapping. It starts at some $(u, v) \in Y$ and iterate $T(u, v)$ until two iterations are identical. The algorithm stops when two iterations are identical ($T(u, v) = (u, v)$). We prove that when the algorithm stops, it must be at a stable payoff. Moreover, we show that all stable payoffs can be reached through by extending the algorithm. The main intuition behing the T-algorithm is similar to the one in Echenique and Oviedo (2006).

T-algorithm:
1. Set $(u^0, v^0) = (u, v)$ where $(u, v) \in Y$. Set $(u^1, v^1) = T(u^0, v^0)$ and $k = 1$.
2. While $(u^k, v^k) \neq (u^{k-1}, v^{k-1})$, do:
(a) set $k = k + 1$
(b) set $(u^k, v^k) = T(u^{k-1}, v^{k-1})$.
3. Set $\tau = (u^k, v^k)$. Stop.

Proposition 2 *If the T-algorithm stops at $\tau \in \mathcal{B}$, then τ is a stable payoff. If (u^k, v^k) is in the set of stable payoffs, for some iteration k of the T-algorithm, then the algorithm stops at $\tau = (u^k, v^k)$.*

Proof. If the algorithm stops at $\tau \in \mathcal{B}$, then $(u^k, v^k) = (u^{k-1}, v^{k-1}) = \tau$. Then, $\tau = T(u^{k-1}, v^{k-1}) = T(\tau)$, so $\tau \in \mathcal{E}(T)$. By Proposition 1, $\tau \in S(\Gamma)$. Moreover, by the formulation of Shapley and Shubik (1972), there is an optimal assignment x such that $(\tau; x) \in C(\Gamma)$. To prove the second part, observe that if (u^k, v^k) is a stable payoff, then (u^k, v^k) is a fixed point of T by Proposition 1. Then the algorithm stops at $\tau = (u^k, v^k)$. ■

We now provide the second algorithm to find all core payoffs. Let

$$(\bar{u}_Y, \underline{v}_Y) = (\max_{w \in W} \alpha_{f_1 w}, ..., \max_{w \in W} \alpha_{f_n w}, 0, ..., 0),$$
$$(\underline{u}_Y, \bar{v}_Y) = (0, ..., 0, \max_{f \in F} \alpha_{f w_1}, ..., \max_{f \in F} \alpha_{f w_n}).$$

Moreover, let e_l^n be the l-th unit vector in $\mathbb{R}^n$, i.e. $e_l^n = (0, ...1, 0, ..., 0) \in \mathbb{R}^n$, where 1 is the l-th element of e_l^n.

Algorithm 2:

1. Set $(u^0, v^0) = (\bar{u}_Y, \underline{v}_Y)$. Set $(u^1, v^1) = T(u^0, v^0)$ and $k = 1$.
2. While $(u^k, v^k) \neq (u^{k-1}, v^{k-1})$, do:
(a) set $k = k + 1$
(b) set $(u^k, v^k) = T(u^{k-1}, v^{k-1})$.
3. Set $\tau = (u^k, v^k)$.
4. Let $\hat{\mathcal{E}} = \tau$. The possible states of the algorithm is Y. Start at state Ω^0 where

$$\Omega^0 = \{(\bar{u}_Y \wedge u^k + e_l^n, 0 \vee v^k - e_m^n), (0 \vee u^k - e_l^n, \bar{v}_Y \wedge v^k + e_m^n)\} \subset Y$$

for all $1 \leq l, m \leq n$. Let the state of the algorithm be Ω. While $\Omega' \neq \emptyset$ do the following subroutine to get a new state Ω'. Then set $\Omega = \Omega'$.

SUBROUTINE: Let $\Omega' = \emptyset$. For each $(u, v) \in \Omega$, run $T(u, v)$. If $T(u, v) = (u, v)$ and $(u, v) \in \mathcal{B}$ add (u, v) to $\hat{\mathcal{E}}$ and add $\{(\bar{u}_Y \wedge u + e_l^n, 0 \vee v - e_m^n), (0 \vee u - e_l^n, \bar{v}_Y \vee v + e_m^n)\} \setminus \hat{\mathcal{E}}$ for all $1 \leq l, m \leq n$ to Ω'.

Theorem 2 *The set $\hat{\mathcal{E}}$ produced by Algorithm 2 coincides with the core payoffs $S(\Gamma)$ of the assignment game.*

Proof. First I prove that the algorithm reaches a fixed point after a finite k number of iterations. Then, we know that $\tau = (u^k, v^k) \in S(\Gamma)$ by Proposition 2. Then I show that $\hat{\mathcal{E}} \subseteq S(\Gamma)$, and $S(\Gamma) \subseteq \hat{\mathcal{E}}$ by using direct proofs.

We want to show that the first part of Algorithm 2, T-algorithm, reaches a fixed point. That is for some finite k, $\tau = (u^k, v^k) = (u^{k-1}, v^{k-1})$. Assume this does not hold for any k. Then, $\{(u^k, v^k)\}$ is an infinite sequence of distinct payoffs in Y. However, there exists a finite number of payoffs that is for all $f \in F$ $u_f \in \{0, 1, ..., \max_{w \in W} \alpha_{fw}\}$ and for all $w \in W$ $v_w \in \{0, 1, ..., \max_{f \in F} \alpha_{fw}\}$, contradicting to the initial assumption. This implies there is $k < \infty$ such that T-algorithm reaches a fixed point.

Now we show that the rest of Algorithm 2 stops after a finite number of steps. Let $M \subseteq Y$ be the collection of states visited by the algorithm. Let $d^1(\Omega)$, where $\Omega \subseteq M$, be the minimum of the Euclidean distance between payoffs in Ω and $(\bar{u}_Y, \underline{v}_Y)$ and $d^2(\Omega)$ be the minimum of the Euclidean distance between payoffs in Ω and $(\underline{u}_Y, \bar{v}_Y)$. If $\Omega = \emptyset$, let $d^1(\Omega) = d^2(\Omega) = 0$. We consider $d^1(\Omega)$ and $d^2(\Omega)$ because if the state is $\{(\bar{u}_Y, \underline{v}_Y), (\underline{u}_Y, \bar{v}_Y)\}$, $\{(\bar{u}_Y, \underline{v}_Y)\}$, or $\{(\underline{u}_Y, \bar{v}_Y)\}$ the next state is $\emptyset$ by the definition of the subroutine. Let Ω' and Ω'' be successive states in the algorithm. It is clear from the definition that $d^1(\Omega') > d^1(\Omega'')$ and $d^2(\Omega') > d^2(\Omega'')$. Since M is a finite set, $d^1(.)$ and $d^2(.)$ takes only a finite number of values. Thus after a finite number of steps the algorithm stops, i.e., $\Omega = \emptyset$.

$\hat{\mathcal{E}} \subseteq S(\Gamma)$. Let $(u, v) \in \hat{\mathcal{E}}$. This implies $(u, v) = T(u, v)$ and $(u, v) \in \mathcal{B}$ by the definition of the algorithm and hence $(u, v) \in \mathcal{E}^{\mathcal{B}}(T)$. By Proposition 1, $\mathcal{E}^{\mathcal{B}}(T) = S(\Gamma)$. Therefore $(u, v) \in S(\Gamma)$ which proves $\hat{\mathcal{E}} \subseteq S(\Gamma)$.

$S(\Gamma) \subseteq \hat{\mathcal{E}}$. Let $(u, v) \in S(\Gamma) = \mathcal{E}^{\mathcal{B}}(T)$. Suppose, by way of contradiction, that $(u, v) \notin \hat{\mathcal{E}}$. This implies $\tau = (u^k, v^k) \neq (u, v)$ and $(u, v) \notin M$ so that the algorithm's states do not contain (u, v). Then either $\tau \succeq_F (u, v)$ or $\tau \preceq_F (u, v)$. Suppose, without loss of generality, $\tau \succeq_F (u, v)$ and $\max_{f \in F}(u^k_f - u_f) = t$. By Lemma 1, there is $w \in W$ such that $v_w - v^k_w = t$. Now we show that $\{(\bar{u}_Y \wedge u + e^n_f, 0 \vee v - e^n_g)\} \notin M$ for all $1 \leq f, g \leq n$. Suppose this is not the case. Then there is a state Ω^c of the algorithm and $a, b \in [1, n]$ such that $(\bar{u}_Y \wedge u + e^n_a, 0 \vee v - e^n_b) \in \Omega^c \subseteq M$. This is only possible if (u, v) is in the previous state $\Omega^{c-1} \subseteq M$ by the definition of the subroutine; a contradiction since we assumed that $(u, v) \notin M$. Using the same argument, we can also conclude that $\{(\bar{u}_Y \wedge u + e^n_f + e^n_h, 0 \vee v - e^n_g - e^n_k)\} \notin M$ for all $1 \leq h, k \leq n$. Repeating the same argument $t-1$ times implies $(\bar{u}_Y \wedge u^k - e^n_l, 0 \vee v^k + e^n_g) \notin M$, which is a contradiction since we have shown that there is $\tau = (u^k, v^k) \in \hat{\mathcal{E}}$ and $(\bar{u}_Y \wedge u^k - e^n_l, 0 \vee v^k + e^n_g) \in \Omega^0 \subseteq M$. This implies $(u, v) \in M$, and hence $(u, v) \in \hat{\mathcal{E}}$. The case where $\tau \preceq_F (u, v)$ is also similar. ■

Now we use the following Example to show the details of the algorithm.

Example 1 [Shapley-Shubik (1972)]. *Let* $\Gamma = \langle \{f_1, f_2, f_3\}, \{w_1, w_2, w_3\}, \alpha \rangle$ *be an assignment game where* α *is*

	w_1	w_2	w_3
f_1	5	8	2
f_2	7	9	6
f_3	2	3	0

.

Algorithm 2 starts at

$$(u^0, v^0) = (8, 9, 3, 0, 0, 0)$$

and does $T(8, 9, 3, 0, 0, 0) = (8, 7, 0, 0, 0, 0)$, $T(8, 7, 0, 0, 0, 0) = (8, 7, 0, 2, 2, 0)$, $T(8, 7, 0, 2, 2, 0) = (6, 6, 0, 2, 2, 0)$, $T(6, 6, 0, 2, 2, 0) = (5, 6, 0, 2, 3, 0)$, $T(5, 6, 0, 2, 3, 0) = (5, 6, 0, 2, 3, 0)$. This implies $\tau = (5, 6, 0, 2, 3, 0)$. Now

$$\Omega^0 = \{(6,6,0,1,3,0), (6,6,0,2,2,0), (6,6,0,2,3,0), (5,7,0,1,3,0), (5,7,0,2,2,0), (5,7,0,2,3,0), (5,6,1,1,3,0), (5,6,1,2,2,0), (5,6,1,2,3,0), (4,6,0,3,3,0), (4,6,0,2,4,0), (4,6,0,2,3,1), (5,5,0,3,3,0), (5,5,0,2,4,0), (5,5,0,2,3,1), (5,6,0,3,3,0), (5,6,0,2,4,0), (5,6,0,2,3,1)\}$$

Note that for all $(u, v) \in \{(5, 6, 1, 1, 3, 0), (4, 6, 0, 2, 4, 0)\} \subset \Omega^0$, $T(u, v) = (u, v)$. Then add $\{(5, 6, 1, 1, 3, 0), (4, 6, 0, 2, 4, 0)\}$ to $\hat{\mathcal{E}}$. The new state is

$$\Omega = \{(5, 6, 1, 1, 3, 0) + (e^3_l, -e^3_m), (5, 6, 1, 1, 3, 0) + (-e^3_l, +e^3_m), (4, 6, 0, 2, 4, 0) + (e^3_l, -e^3_m), (4, 6, 0, 2, 4, 0) + (-e^3_l, e^3_m)\} \setminus \{(5, 6, 0, 2, 3, 0)\}.$$

For all $(u, v) \in \{(4, 6, 1, 1, 4, 0), (4, 5, 0, 2, 4, 1), (3, 6, 0, 2, 5, 0)\} \subset \Omega$, $T(u, v) = (u, v)$. Then add $\{(4, 6, 1, 1, 4, 0), (4, 5, 0, 2, 4, 1), (3, 6, 0, 2, 5, 0)\}$ to $\hat{\mathcal{E}}$. The new state is

$$\Omega' = \{(4, 6, 1, 1, 4, 0) + (e^3_l, -e^3_m), (4, 6, 1, 1, 4, 0) + (-e^3_l, +e^3_m), (4, 5, 0, 2, 4, 1) + (e^3_l, -e^3_m), (4, 5, 0, 2, 4, 1) + (-e^3_l, +e^3_m), (3, 6, 0, 2, 5, 0) + (e^3_l, -e^3_m), (3, 6, 0, 2, 5, 0) + (-e^3_l, +e^3_m)\} \setminus \hat{\mathcal{E}}.$$

It is only the case that for $(3,5,0,2,5,1) \in \Omega'$, $T(3,5,0,2,5,1) = (3,5,0,2,5,1)$. Then add $(3,5,0,2,5,1)$ to $\hat{\mathcal{E}}$. The new state is

$$\Omega'' = \{(3,5,0,2,5,1) + (e_l^3, -e_m^3), (3,5,0,2,5,1) + (-e_l^3, +e_m^3)\} \setminus \hat{\mathcal{E}}.$$

Note that there is not any $(u,v) \in \Omega''$ such that $T(u,v) = (u,v)$. Then the new state is $\emptyset$. This implies the algorithm stops and the core of the assignment game is

$$\hat{\mathcal{E}} = \{(5,6,0,2,3,0), (5,6,1,1,3,0), (4,6,0,2,4,0), (4,6,1,1,4,0), (4,5,0,2,4,1), (3,6,0,2,5,0), (3,5,0,2,5,1)\}.$$

5 Formulation with core outcomes

It would be nice to find a construction such that fixed points will directly provide the core outcomes. However unlike the assignment literature without money, it is not possible to work with core outcomes in this setup. In the rest of the paper, we define a reasonable construction which can work with outcomes. Then, we provide examples to show that this type of formulation is not possible.

Let π be a **pre-assignment** if $\pi : F \cup W \to F \cup W$ such that $\pi(f) \in W \cup \{f\}$ for all $f \in F$, and $\pi(w) \in F \cup \{w\}$ for all $w \in W$. Let Π be the set of all pre-assignment vectors. Define a map $T' : Y \times \Pi \to Y \times \Pi$ such that

$$T'_f((u,v);\pi(f)) = ((\max U_f(u,v)) \vee 0; w) \textit{ where } w \in \textit{argmax } (\alpha_{fw} - v_w) \quad \forall f \in F,$$

and

$$T'_w((u,v);\pi(w)) = ((\max V_w(u,v); f)) \vee 0 \textit{ where } f \in \textit{argmax } (\alpha_{fw} - u_f) \quad \forall w \in W.$$

Then we could show that the fixed points of T' is equivalent to the core. However, this type of formulation is not possible in this framework since there might be more than one optimal assignment and different (pre)assignments might correspond to same payoffs. Then, fixed point of T' may fail to induce an assignment. On the other hand, by using our formulation core payoffs can always be found and core outcomes will be equal to the Cartesian product of the fixed points and the set of optimal assignments which is constructed.

Example 2 (Shapley-Shubik (1972)) *Let* $\Gamma = \langle\{f_1, f_2, f_3\}, \{w_1, w_2, w_3\}, \alpha\rangle$ *be an assignment game where* α *is*

	w_1	w_2	w_3
f_1	*0*	*2*	*0*
f_2	*2*	*0*	*2*
f_3	*0*	*2*	*0*

There are four optimal assignments given by

$$\mathcal{X} = \{(0,1,0;1,0,0;0,0,1),(0,1,0;0,0,1;1,0,0),(0,0,1;1,0,0;0,1,0),(1,0,0;0,0,1;0,1,0)\}.$$

with value $\sum_{(f,w)\in F\times W} \alpha_{fw}x_{fw} = 4$. The core of the game given by

$$C(\Gamma) = (0,2,0,0,2,0) \times \mathcal{X}.$$

Moreover, $((0,2,0,0,2,0);\pi)$ where $\pi(f_1) = w_1, \pi(f_2) = w_3, \pi(f_3) = w_2, \pi(w_1) = f_3, \pi(w_2) = f_1, \pi(w_3) = f_3$ is a fixed point of T' with appropriate tie breaking rule but π is not an assignment. Hence $((0,2,0,0,2,0);\pi) \notin C(\Gamma)$.

Using Example 1, we can show that a construction like T' will not work even though there is a unique optimal assignment. In the example, there exists a unique optimal assignment given by

$$\mathcal{X} = \{(0,1,0;0,0,1;1,0,0)\}$$

with value $\sum_{(f,w)\in F\times W} \alpha_{fw}x_{fw} = 16$. It is easy to see that $(3,5,0,2,5,1) \in S(\Gamma')$. Moreover, $((3,5,0,2,5,1);\pi)$ where $\pi(f_1) = w_1, \pi(f_2) = w_3, \pi(f_3) = w_2, \pi(w_1) = f_3, \pi(w_2) = f_1, \pi(w_3) = f_2$ is a fixed point of T' with appropriate tie breaking rule but π is not an assignment. Hence $((3,5,0,2,5,1);\pi) \notin C(\Gamma')$.

6 Final remarks

In our formulation, we work with payoffs and construct optimal assignments rather than directly working with outcomes. The main reason for that is different (pre) assignments might lead to a same payoff structure and the mapping defined on feasible outcomes may fail to induce an assignment. Moreover, defining a partial order on the Cartesian product of the payoffs and (pre)assignments is a problem.

Such a formulation (if it is not impossible) which works also with outcomes, seems to be an important follow-up to our work. The extension of our algorithm to many-to-one and many-to-many assignment games will be a subject of our future work.

References

[1] Abizada, A. (2016), "Stability and incentives for college admissions with budget constraints," *Theoretical Economics* **11**, 735-756.

[2] Adachi, H. (2000), "On a Characterization of Stable Matchings," *Economic Letters* **68**, 43-49.

[3] Afacan, M. O. (2013), "Application fee manipulations in matching markets," *Journal of Mathematical Economics* **49**, 446-453.

[4] Echenique, F. and J. Oviedo (2004), "Core many-to-one matchings by fixed point methods," *Journal Economic Theory* **115**, 358-376.

[5] Echenique, F. and J. Oviedo (2006), "A theory of stability in many-to-many matching markets," *Theoretical Economics* **1**, 233-273.

[6] Echenique, F. and M. Yenmez (2007), "A Solution to Matching with Preferences Over Colleagues," *Games and Economic Behavior*, **59**, 46-71.

[7] Kucuksenel, S. (2011), "Core of the Assignment Game via Fixed Point Methods," *Journal of Mathematical Economics*, **47**, 72-76.

[8] Rochford, S.C. (1984), "Symmetrically Pairwise-Bargained Allocations in an Assignment Market," *Journal Economic Theory* **34**, 262-281.

[9] Roth, A.E. and M. Sotomayor (1990), "*Two sided matching: A Study in Game-Theoretic Modelling and Analysis,*" Cambridge University Press, Cambridge. Econometric Society Monograph Series.

[10] Shapley, L.S. and J. Shubik (1972), "The Assignment Game I: The Core," *International Journal of Game Theory* **1**, 111-130.