Q.1 a. Prove that if $D(G)$ is any representation of a finite group G on an inner product space $\boldsymbol{V}$ and $x\,,y \in \boldsymbol{V}$ then

$$(x\,,y) = \sum_{g\in G}\{D(g)x|D(g)y\}\,,$$

defines a new sac;r product on $\boldsymbol{V}$.

Q.2 a. Consider the dihedral-3 group D_3 of order 6. Let $\boldsymbol{V}$ be a 2-dimensional vector space spanned by the the vectors $\hat{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\boldsymbol{e}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

a. Write down a matrix representation of D_3 on $\boldsymbol{V}$ with respect to the Cartesian basis given above.

b. Is the representation that you have found in part a.) reducible or irreducible ?

c.Give a unitary irreducible representation od $D_3 \simeq S_3$ on this vector space $\boldsymbol{V}$.

Q.3 Consider the 2-dimensional function space $\boldsymbol{V}^*$ consisting of the polynomials

$$f(x,y) = ax + by\,, \quad a\,,b \in \mathbb{C}$$

As we have discussed in the class, these functionals form a representation of D_2 and R_2. Note also that we can write

$$f(x,y) = \langle a\,,b|x\,,y\rangle\,,$$

in a self-evident notation, where the "bra"'s $\langle a\,,b|$ stand abstract as an element of $\boldsymbol{V}^*$.

a.) Identify and write down the invariant subspaces and irreducible representations (IRR)'s of D_2 on $\boldsymbol{V}^*$.

b.) Identify and write down the invariant subspaces and IRR's of R_2 on $\boldsymbol{V}^*$.

Q.4 Consider the group of translations in 1-dimensions. The elements of this group can be denoted as $T(a)$ such that $T(a)\boldsymbol{x} = \boldsymbol{x} + a$. Find a n-dimensional representation on the space of functionals of x in n-dimensions, i.e. in the n-dimensional space of polynomials of x of degree $n-1$. Show for instance in the case of $n = 3$ that, $T(a)T(b) = T(a+b)$.