Q.1 a. Prove Cayley's theorem, which states that every group G of order n is isomorphic to a subgroup of the symmetric group S_n.

b. Show that if the order n of the group is a prime number then it must be isomorphic to C_n. Conclude that as long as n is prime, there exist only one group of this order.

Q.2 Prove Lagrange's theorem, stating that the order of a finite group must be an integer multiple of the order of any of its subgroups.

Q.3 a. Show that if $G = H_1 \otimes H_2$ then we have the isomorphisms: $G/H_1 \simeq H_2$ and $G/H_2 \simeq H_1$.

b. Show that if H is an invariant subgroup of G and $G/H \simeq H'$, then it does NOT in general follow that $G = H \otimes H'$.

Q.4 Consider the symmetric group S_4.

a.) Find all of its conjugacy classes.

b.) Find all the invariant subgroups of S_4.

c.) In part b.) you should have found two invariant subgroups H_1 and H_2. Show that $S_4/H_1 \simeq C_2$ and $S_4/H_2 \simeq S_3$.

Hint: In part b.) Use Lagrange's theorem and the fact that invariant subgroups can contain only complete classes.

Q.5 Consider the dihedral group D_4 of order 8. This the symmetry group of a square consisting of identity, c.c.w rotations about the centre of a square by the angles $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and reflections about vertical, horizontal and diagonal axis.

a.) Enumerate the group elements, find all of its conjugacy classes.

b.) Find all the subgroups and the invariant subgroups.

c.) In part b.) you should have found 10 subgroups of which 4 of them are invariant excluding the trivial invariant subgroups. Find the corresponding quotient groups for these invariant subgroups.

d.) Convince yourself that a possible way to present the group D_4 is given by

$$\langle (a, x) \big| a^4 = x^2 = e\,, xax^{-1} = a^{-1} \rangle\,,$$

where a stands for any of the three rotations and x is reflection. Noting also that $x^{-1} = x$, we can also $xax = a^{-1}$.

In this notation the group elements can be listed as $\{e, a, a^2, a^3, x, ax, a^2x, a^3x\}$. Note the similarity and the generalisation that follows from the way we present the group C_4.

In this notation how do you express the groups D_2, D_3 and D_n.