ZETA FUNCTIONS AND L-SERIES

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Abstract. These are the lecture notes of "Algebraic Numbers and L-functions Workshop" held in Adrasan, September 2010. I would like to thank the organizers again for inviting me as a speaker.

1. Day

- Real Analysis: Several techniques to extend a function from a given set to a larger one (unless vertical asymptotes).
  - Continuity ✓
  - Smoothness ✓
- Complex Analysis: Rigid properties make it difficult to extend a function smoothly.

Example 1.1. Let $f(z)$ be the function given by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

Its radius of convergence is $R = 1$. If $z = re^{i\theta}$, then $|f(z)| \to \infty$ as $r \to 1$. Therefore $f(z)$ cannot be continued analytically past the unit disc.

Suppose f and g are holomorphic in a region Ω. If $f(z) = g(z)$ in some non-empty open subset of Ω, then the equality holds throughout Ω. Therefore analytic continuation is unique if it exists.

1.1. The Gamma Function.

For $s > 0$, the gamma function is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt.$$

Note that $\Gamma(1) = 1$.

Lemma 1.2. *The function $\Gamma(s)$ extends to an analytic function in the half-plane $\Re(s) > 0$ and given by the integral formula above.*

Proof. Let δ, M be positive real numbers and let $S_{\delta,M} = \{s \in \mathbf{C} : \delta < \Re(s) < M\}$. For each $s \in S_{\delta,M}$, define

$$F_\epsilon(s) = \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1} dt.$$

Set $\sigma = \Re(s)$. We have

$$|\Gamma(s) - F_\epsilon(s)| \leq \int_0^{\epsilon} e^{-t} t^{\sigma-1} dt + \int_{1/\epsilon}^{\infty} e^{-t} t^{\sigma-1} dt.$$

One can show that both integrals converge zero uniformly. The result follows easily. □

The function $\Gamma(s)$ satisfies

$$\Gamma(s+1) = s\Gamma(s)$$

for $\Re(s) > 0$. This is a *functional equation* and can be proved using integration by parts. A corollary of this fact is $\Gamma(n+1) = n!$.

Theorem 1.3. *The function $\Gamma(s)$, initially defined for $\Re(s) > 0$, has an analytic continuation to a meromorphic function on $\mathbf{C}$ whose only singularities are simple poles at the negative integers $s = 0, -1, -2, \ldots$ and the residue of Γ at $s = -n$ is $(-1)^n/n!$.*

Proof. For $\Re(s) > -1$, define

$$F_1(s) = \frac{\Gamma(s+1)}{s}.$$

This is a meromorphic function in the half plane $\Re(s) > -1$ with a simple pole at $s = 0$. Observe that $F_1(s) = \Gamma(s)$ on $\Re(s) > 0$. We can continue extending $\Gamma(s)$ in this fashion. In general

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots(s+m-m).}$$

Successive applications of functional equation show that $F_m(s) = \Gamma(s)$ on $\Re(s) > 0$. Therefore we have obtained the desired continuation of Γ. $\square$

Question 1.4. Can we solve $\Gamma(s) = 0$ or $1/\Gamma(s) = 0$?

The theorem above already gives a complete answer the to latter question. To answer the other one we use the fact

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

which is valid for all $s \in \mathbf{C}$. It is easy to see that $\Gamma(s)$ is zero-free. This equation also reveals the symmetry of Γ about the line $\Re(s) = 1/2$. In particular $\Gamma(1/2) = \sqrt{\pi}$.

1.2. **The Riemann Zeta Function.** For $s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Set $\sigma = \Re(s)$. Then $|n^{-s}| = n^{-\sigma}$. If $\sigma > \delta + 1 > 1$, then $\zeta(s)$ is uniformly bounded by $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$. Therefore $\zeta(s)$ converges uniformly on $\Re(s) > 1 + \delta$ and we have the following result.

Lemma 1.5. *The series defining $\zeta(s)$ converges for $\Re(s) > 1$ and the function $\zeta(s)$ is holomorphic in this half-plane.*

Theorem 1.6. *The function $\zeta(s)$ admits a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$ with residue 1. It satisfies the functional equation*

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Proof. In order to relate $\zeta(s)$ and $\Gamma(s)$ suitably, we use the substitution $t \mapsto \pi n^2 t$. We have

$$\pi^{-s}\Gamma(s)\frac{1}{n^{2s}} = \int_0^{\infty} e^{-\pi n^2 t} t^s \frac{dt}{t}.$$

Summing over all positive integers n, we get

$$\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 t} t^s \frac{dt}{t}.$$

The theta series $\theta(t) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 t}$ naturally appears in this expression. The completed Riemann zeta function is defined by

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It is clear from above equation that

$$Z(2s) = \frac{1}{2}\int_0^\infty (\theta(it) - 1)t^s \frac{dt}{t}.$$

Using Mellin transform and the fact $\theta(-1/t) = \sqrt{(t/i)}\theta(t)$, one can show that

$$Z(s) = Z(1-s).$$

The result follows from Legendre's duplication formula

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$$

together with the functional equation of $Z(s)$. □

Example 1.7. It is a well-known fact in complex analysis that

$$\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We can use this together with the above theorem to obtain

$$\begin{aligned}\zeta(-1) &= 2(2\pi)^{-2}\Gamma(2)\cos(\pi)\zeta(2)\\ &= -1/12.\end{aligned}$$

2. Day

The Riemann zeta function can be written as an infinite product

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

where p runs through prime numbers. This is called Euler's identity and valid for $\Re(s) > 1$. In order to prove this identity, one should start with the geometric series expansion of $1/(1-x)$.

An immediate consequence of Euler's identity is that $\zeta(s)$ does not vanish when $\Re(s) > 1$. In order to investigate the zeros of $\zeta(s)$, let us use $Z(s) = Z(1-s)$ and write

$$\zeta(s) = \pi^{(s-1)/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1-s).$$

Observe that for $\Re(s) < 0$, the following are true:

- $\zeta(1-s)$ has no zeros because $\Re(1-s) > 1$.
- $\Gamma((1-s)/2)$ is zero-free.
- $1/\Gamma(s/2)$ has zeros at even negative integers.

Theorem 2.1. *The only zeros of $\zeta(s)$ outside the strip $0 \leq \Re(s) \leq 1$ are at the negative integers $-2, -4, -6, \ldots$.*

The region $0 \leq \Re(s) \leq 1$ is called the critical strip. The Riemann Hypothesis claims that the zeros of $\zeta(s)$ in the critical strip lie on the line $\Re(s) = 1/2$.

2.1. **Prime Number Theorem.** Euler discovered a deep connection between analytical methods and arithmetic properties. For example one can use the divergence of the series $\sum_p 1/p$ to show that there are infinitely many primes.

One can also try to understand the distribution of primes by analytical methods. For this purpose, define $\pi(x)$ to be the number of primes less than or equal to x. Legendre and Gauss observed that

$$\pi(x) \sim \frac{x}{\log(x)}.$$

In other words, the limit of their quotient goes to 1 as x goes to ∞.

This interesting fact can be proved by detailed manipulations of the Riemann zeta function near the line $\Re(s) = 1$. A key point is to show that $\zeta(s)$ is zero-free on this line.

2.2. **Special Values of $\zeta(s)$.** In complex calculus, we can use contour integrals to get

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}.$$

It is not hard to guess that there is a pattern. It turns out that these values can be obtained by Bernoulli numbers B_k. They are defined as follows

$$\frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The fundamental relation between $\zeta(s)$ and Bernoulli numbers is given by the following result.

Theorem 2.2. *For every integer $k > 0$, one has*

$$\zeta(1-k) = -\frac{B_k}{k}.$$

This theorem can be proved by starting Cauchy's integral formula. Define $F(z) = \frac{ze^z}{e^z - 1}$. This is a meromorphic function with only poles at $z = 2\pi i\nu$ where ν is a non-zero integer. We have

$$\mathrm{Res}_{z=0} F(z) z^{-k-1} = \frac{1}{2\pi i} \int_{|z|=\epsilon} F(z) z^{-k-1} dz.$$

It is easy to see that the left hand side is $B_k/k!$. In order prove the theorem, we need to show that the integral on the right equals $-\zeta(1-k)/(k-1)!$. This can be done by manipulating the path of integration $|z| = \epsilon$ by including the infinity.

Let us use this theorem with the functional equation of $\zeta(s)$ at $s = 2k$ and obtain

$$\zeta(1-2k) = 2(2\pi)^{-2k}\Gamma(2k)\cos(\pi k)\zeta(2k).$$

This equality justifies the pattern observed above. The values of $\zeta(2k)$ for $k = 1, 2, \ldots$ are given by

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

2.3. **Dedekind Zeta Function.** The Dedekind zeta function is concerned with number fields. A number field K is a finite extension of rational numbers $\mathbf{Q}$. It has a special subring of integral elements denoted by $\mathcal{O}_K$. The discriminant d_K and the class number h_K are important invariants of a number field.

Example 2.3. If $K = \mathbf{Q}(\sqrt{-23})$ and $\alpha = (\sqrt{-23}+1)/2$, then $\mathcal{O}_K = \mathbf{Z} \oplus \alpha\mathbf{Z}$. Observe that α is a root of the polynomial $x^2 - x + 6$. The discriminant is obtained by

$$d_K = \det \begin{bmatrix} 1 & \alpha \\ 1 & \bar{\alpha} \end{bmatrix}^2 = -23$$

The ring of integers $\mathcal{O}_K$ is not a unique factorization domain in general. For example we have the following decompositions of $6 = 2 \cdot 3 = \alpha\bar{\alpha}$. The elements $2, 3, \alpha, \bar{\alpha}$ are all distinct and irreducible. Even though $\mathcal{O}_K$ is not a UFD, there is unique factorization in terms of ideals. This property holds since $\mathcal{O}_K$ is a Dedekind domain. We have

$$(6) = \mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_3\bar{\mathfrak{p}}_3$$

where $\mathfrak{p}_2 = (2, \alpha)$ and $\mathfrak{p}_3 = (3, \alpha)$.

An ideal $I \subset \mathcal{O}_K$ is principal if there exists $w \in \mathcal{O}_K$ such that $I = w\mathcal{O}_K = (w)$. We define the class group

$$\mathrm{Cl}(K) = \frac{\text{Fractional ideals of } \mathcal{O}_K}{\text{Principal ideals of } \mathcal{O}_K}.$$

It is a well known fact that this group is finite and its order h_K is called the class number.

Given a non-zero integral ideal $\mathfrak{a} \subset \mathcal{O}_K$, the norm $\mathcal{N}(\mathfrak{a})$ is the number of elements in $\mathcal{O}_K/\mathfrak{a}$. The Dedekind zeta function of the number field K is defined by the series

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^s}.$$

It is easy to see that if $K = \mathbf{Q}$, then $\zeta_K(s) = \zeta(s)$. In other words the Dedekind zeta function is a generalization of Riemann zeta function.

Theorem 2.4. *The series $\zeta_K(s)$ converge absolutely and uniformly in the domain $\Re(s) \geq 1 + \delta > 1$ and one has the Euler product*

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}}$$

where $\mathfrak{p}$ runs through prime ideals of $\mathcal{O}_K$.

The Dedekind zeta function has a strong connection with the arithmetic of the corresponding number field. To illustrate this let us consider $K = \mathbf{Q}(\sqrt{d_K})$, a quadratic number field of discriminant d_K. Define

$$\chi(p) = \begin{cases} 0 & \text{if } (d_K, p) \neq 1 \\ 1 & \text{if } d_K = \square \pmod{p} \\ -1 & \text{if } d_K \neq \square \pmod{p}. \end{cases}$$

This is an example of a Dirichlet character, a multiplicative homomorphism from $(\mathbf{Z}/d_K Z)^*$ to $\mathbf{C}^*$. The ideal $p\mathcal{O}_K$ decomposes in $\mathcal{O}_K$ as follows:

- If $\chi(p) = 0$, then $p\mathcal{O}_K = \mathfrak{p}^2$ (p ramifies).

- If $\chi(p) = 1$, then $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ (p splits).
- If $\chi(p) = -1$, then $p\mathcal{O}_K = \mathfrak{p}$ (p inerts).

Starting with Euler product of the Dedekind zeta function and organizing each term by its behavior in the extension $K/\mathbf{Q}$, we obtain the following

$$\begin{aligned}
\zeta_K(s) &= \prod_{\mathfrak{p}} \frac{1}{1-\mathcal{N}(\mathfrak{p})^{-s}} \\
&= \prod_{\chi(p)=0} \frac{1}{1-\mathcal{N}(\mathfrak{p})^{-s}} \left(\prod_{\chi(p)=1} \frac{1}{1-\mathcal{N}(\mathfrak{p})^{-s}} \right)^2 \prod_{\chi(p)=-1} \frac{1}{1-\mathcal{N}(\mathfrak{p})^{-2s}} \\
&= \prod_p \frac{1}{1-p^{-s}} \prod_p \frac{1}{1-\chi(p)p^{-s}}
\end{aligned}$$

It is easy to see that the former product equals $\zeta(s)$ by Euler's identity. The second product turns out to be $L(s,\chi)$, a Dirichlet L-function. It is an astonishing fact that

$$L(1,\chi) = \frac{2^{r_1}(2\pi)^{r_2} h_K \mathrm{Reg}(K)}{W_K\sqrt{d_K}}.$$

This fact follows from the class number formula and will be investigated tomorrow.

3. Day

The Dedekind zeta function $\zeta_K(s)$ has an analytic continuation to $\mathbf{C} - \{1\}$. At $s = 1$, it has a simple pole with residue

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K \mathrm{Reg}(K)}{W_K\sqrt{d_K}}.$$

This is called the class number formula and includes several important invariants of the number field K. The number of real embeddings K into real numbers is r_1 and the number of complex embedding pairs is r_2. It is easy to see that $[K:\mathbf{Q}] = r_1 + 2r_2$. It is a well known fact that

$$\mathcal{O}_K^* \cong \mu_K \times \mathbf{Z}^r$$

where $r = r_1 + r_2 - 1$. Here μ_K is the group of roots of unity in the number field K. Its order is denoted by W_K. Computing the invariant $\mathrm{Reg}(K)$ is hard since it requires finding generators to the unit group $\mathcal{O}_K^*$. This difficulty prevents us from using this theorem to compute the class number h_K if r is large.

3.1. Dirichlet L-Series.

Let $K/\mathbf{Q}$ be a finite Abelian extension. Kronecker-Weber Theorem implies that K is a subfield of a cyclotomic number field $\mathbf{Q}(\zeta_m)$ for some m. Here ζ_m is a primitive m-th root of unity. It is an easy exercised to show that

$$G = \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^*.$$

Let a be an integer coprime to m. Then the canonical isomorphism (from right to left) is given by $a \mapsto \sigma_a$ where $\sigma_a(\zeta_m) = \zeta_m^a$. A Dirichlet character is a multiplicative homomorphism

$$\chi : (\mathbf{Z}/m\mathbf{Z})^* \to \mathbf{C}^*.$$

If a Dirichlet character cannot be factored through $(\mathbf{Z}/m\mathbf{Z})^* \to (\mathbf{Z}/m'\mathbf{Z})^* \to \mathbf{C}^*$ for some $m' < m$ dividing m then it is called primitive. The isomorphism above enables us to identify characters of G with Dirichlet characters modulo m. There is one-to-one correspondence with subgroups X of Dirichlet characters and subfields of $\mathbf{Q}(\zeta_m)$.

Example 3.1. Let X be the subgroup of Dirichlet characters with $\chi(-1) = 1$. Note that the map $\zeta_m \mapsto \zeta_m^{-1}$ is the complex conjugation. This means that complex conjugation act trivially on the corresponding subfield of $\mathbf{Q}(\zeta_m)$. Therefore it must be $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$, the real m-th cyclotomic field.

Set $\chi(n) = 0$ if $(n, m) \neq 1$. The Dirichlet L-series is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

for $\Re(s) > 1$. Note that $L(s, \mathbf{1}) = \zeta(s)$. It has an analytic continuation to the whole complex plane except when $\chi = \mathbf{1}$ and $s = 1$. For quadratic fields we have found that $\zeta_K(s) = \zeta(s)L(s, \chi)$. If $K/\mathbf{Q}$ is an Abelian extension, then in general

$$\zeta_K(s) = \prod_{\chi \in X} L(s, \chi)$$

where each χ is primitive. Class number formula implies that

$$\prod_{\chi \in X'} L(1, \chi) = \frac{2^{r_1}(2\pi)^{r_2} h_K \mathrm{Reg}(K)}{W_K \sqrt{|d_K|}}.$$

where X' is the set of all non-trivial characters attached to K. Now we illustrate this formula by applying it to the quadratic fields

Example 3.2 (Imaginary quadratic)**.** Let $K = \mathbf{Q}(\sqrt{d_K})$ with $d_K < 0$. We have $r_1 = 0$ and $r_2 = 1$. Since $r = 0$, there is no fundamental unit in K. Therefore $\mathrm{Reg}(K) = 1$. It is easy to show that

$$W_K = \begin{cases} 6 & \text{if } d_K = -3 \\ 4 & \text{if } d_K = -4 \\ 2 & \text{otherwise.} \end{cases}$$

and the class number formula gives us

$$h_K = \frac{L(1, \chi) W_K \sqrt{|d_K|}}{2\pi}.$$

In particular, pick $d_K = -4$. The corresponding character of $K = \mathbf{Q}(i)$ is characterized by n mod (d_K). If $n = \pm 1 \pmod 4$, then $\chi(n) = \pm 1$ respectively. We set $\chi(n) = 0$ if a is even. We compute

$$\begin{aligned} L(1, \chi) &= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \\ &= \arctan(1) \\ &= \pi/4. \end{aligned}$$

From this, we obtain that $h_K = 1$ and therefore $\mathcal{O}_K = \mathbf{Z}[i]$ is a principal ideal domain.

The series $L(1,\chi)$ is conditionally convergent and not very suitable to compute. A nice transformation for $d_K < -4$ gives the class number as a finite sum

$$h_K = \frac{1}{d_K} \sum_{1 \le r \le |d_K|} r\chi(r).$$

Example 3.3 (Real quadratic)**.** Let $K = \mathbf{Q}(\sqrt{d_K})$ with $d_K > 0$. We have $r_1 = 2$ and $r_2 = 0$. Finding a fundamental unit u, one can write $\mathcal{O}_K^* = \{\pm u^k : k \in \mathbf{Z}\}$ since $r = 1$. It is easy to see that $\mathrm{Reg}(K) = \log|u|$ and $W_K = 2$. The class number formula gives us

$$h_K = \frac{L(1,\chi)\sqrt{d_K}}{2\mathrm{Reg}(K)}.$$

In particular, pick $K = \mathbf{Q}(\sqrt{2})$. One can show that $\mathcal{O}_K = \mathbf{Z} \oplus \sqrt{2}\mathbf{Z}$ and

$$d_K = \det \begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix}^2 = 8.$$

The corresponding character of K is characterized by $n \pmod 8$. We have $\chi(n) = 1$ if $n = \pm 1 \pmod 8$ and $\chi(n) = -1$ for other odd integers. Using this, we compute

$$\begin{aligned} L(1,\chi) &= \left(\frac{1}{1} - \frac{1}{3} - \frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15}\right) + \dots \\ &\cong 0.623 \end{aligned}$$

Putting this in the above equation, we find that

$$h_K = \frac{L(1,\chi)\sqrt{8}}{2\log(\sqrt{2}-1)} = 1.$$

It is not easy to find fundamental units generating $\mathcal{O}_K^*$ in general. For example if $K = \mathbf{Q}(\sqrt{151})$, then

$$\pm u = 140634693\sqrt{151} - 1728148040.$$

As r gets large, finding a complete set of generators for the unit group becomes harder. Therefore the class number formula is usually not practical for direct computation of class number.

3.2. **Arithmetic Progression.**

Theorem 3.4. *Every arithmetic progression*

$$a, a \pm m, a \pm 2m, a \pm 3m, \dots$$

with $(a,m) = 1$ *contains infinitely many prime numbers.*

Proof. Let χ be a Dirichlet character modulo m. Using Euler product of Dirichlet L-series for $\Re(s) > 1$ together with Maclaurin series of $-\log(1-x)$, we obtain

$$\begin{aligned} \log L(s,\chi) &= -\sum_p \log(1 - \chi(p)p^{-s}) \\ &= \sum_p \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}}. \end{aligned}$$

Collecting all terms with $m \geq 2$, we get

$$\log L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + g_\chi(s)$$

It is easy to show that $g_\chi(s)$ is holomorphic at $s = 1$. Summing over all Dirichlet characters modulo m, we establish the equality

$$\sum_\chi \chi(a^{-1}) \log L(s,\chi) = \sum_{p=a(m)} \frac{\phi(m)}{p^s} + g(s).$$

It is a well known fact for $\chi \neq \mathbf{1}$ that $L(s,\chi)$ is holomorphic at $s = 1$. Moreover it does not vanish at that point. Therefore the left hand side tends to infinity as s approaches 1, due to the pole of Riemann zeta function $\zeta(s) = L(s,\mathbf{1})$. The function $g(s)$ equals $\sum_\chi g_\chi(s)$ and it is holomorphic at $s = 1$. We find

$$\lim_{s\to 1} \sum_{p=a(m)} \frac{\phi(m)}{p^s} = \infty.$$

The sum cannot consist of finitely many terms. □

4. Day

Last time, we have shown that the arithmetic progression can be proved using the fact that $L(1,\chi) \neq 0$, where χ is nontrivial. This value plays a special role in algebraic number theory. It can be computed easily and enables us to obtain important results of the class number of cyclotomic extensions and their subfields. In order to describe the values of $L(1,\chi)$, one should use the generalized Bernoulli numbers $B_{n,\chi}$. Let χ be a character modulo m which is primitive. By definition

$$\sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} = \sum_{a=1}^{m} \frac{\chi(a) t e^{at}}{e^{mt} - 1}.$$

The values $L(1,\chi)$ can be classified with respect to the value of χ at -1. If $\chi(-1) = -1$ then χ is called an odd character and we have

$$L(1,\chi) = \pi i \frac{\tau(\chi)}{m} B_{1,\bar{\chi}}.$$

Otherwise χ is called even. In such a case, we have

$$L(1,\chi) = \frac{\tau(\chi)}{m} \sum_{a=1}^{m} \bar{\chi}(a) \log|1 - \zeta_m^a|.$$

unless χ is trivial.

Let us consider the cyclotomic field $K = \mathbf{Q}(\zeta_m)$. Assume that $m \neq 2 \pmod 4$. Otherwise we can divide m by 2, and still obtain the same field. The real cyclotomic field $K^+ = \mathbf{Q}(\zeta_m + \zeta_m^{-1})$ is totally real, in other words $r_1 = [K^+ : \mathbf{Q}] = \phi(m)/2$. On the other hand, the embeddings of K are all complex, in other words $[K : \mathbf{Q}] = 2r_2 = \phi(m)$. Therefore these number fields have the same unit rank $r = \phi(m)/2 - 1$.

It is a well known fact that $\mathrm{Cl}(K^+)$ injects into $\mathrm{Cl}(K)$. This enables us to define an integer h^-, by setting

$$h = h^+ \cdot h^-$$

where $h = h_K$ and $h^+ = h_{K^+}$. Note that $r = \phi(m)/2-1$ is also the number of even characters modulo m. Applying class number formula twice and "cancelling" the values of these even characters, we obtain the following

$$h^- = Qm \prod_{\chi \text{ odd}} \left(\frac{-1}{2} B_{1,\chi} \right).$$

Here Q is equal to 1 if m is a power of a prime and 2 otherwise. One can use this equality and detailed approximations to get $h^- > 1$ if $\phi(m) \geq 220$. Using this, one can easily answer the class number one problem for cyclotomic extensions.

4.1. **Unit-Index Theorem.** The previous part explains how to find the minus part of the class number quite well. Computing generalized Bernoulli numbers, we can easily obtain h^-. In order to compute h, we also need the plus part, namely the class number of the underlying real cyclotomic field. However h^+ is extremely hard to compute. For example the class number of p-th real cyclotomic field is not known for any prime $p > 113$. Cyclotomic units are extremely helpful to understand h^+. Given an integer a coprime to m, define

$$\eta_a = \frac{1 - \zeta_m^a}{1 - \zeta_m} \zeta_m^{(1-a)/2}.$$

Note that η_a is an algebraic integer and invariant under complex conjugation. Therefore it is an element of $\mathcal{O} = \mathcal{O}_{K^+}$. Moreover it is of norm 1 and this implies that η_a is in $\mathcal{O}^*$. Let $\mathcal{C}$ be the multiplicative $\mathbf{Z}[G^+]$-module generated by these units where $G^+ = \mathrm{Gal}(K^+/\mathbf{Q})$.

Theorem 4.1. *Let $m = p^s$ be a power of a prime. Then*

$$h^+ = [\mathcal{O}^* : \mathcal{C}].$$

Proof. A crucial property of regulators implies that it is enough to show

$$\mathrm{Reg}(\{\eta_a\}) = \mathrm{Reg}(\mathcal{O}^*) \cdot h^+.$$

Using the definition of $\mathrm{Reg}(\{\eta_a\})$ and then a determinant trick, we obtain

$$\begin{aligned} \mathrm{Reg}(\{\eta_a\}) &= \pm \det[\log |\eta_a^\tau|]_{a,\tau \neq 1} \\ &= \pm \prod_{\chi \neq \mathbf{1}} \sum_{\sigma \in G^+} \chi(a) \log |(1-\zeta)^\sigma| \end{aligned}$$

Once we relate each sum above to $L(1, \chi)$, it is easy to see that their product is $\mathrm{Reg}(\mathcal{O}^*) \cdot h^+$ by the class number formula. An important step in the proof is to use primitive characters. It is possible if we pick $m = p^s$, a power of a prime. □

4.2. **Kronecker's Jugendtraum.** This is also known as Hilbert's 12th problem. It concerns with functions which generate Abelian extensions of a given number field. Currently there are only two complete answers:

- If $K = \mathbf{Q}$, then use $\exp(z)$. This is a consequence of Kronecker-Weber theorem.
- If $K = \mathbf{Q}(\sqrt{d_K})$ with $d_K < 0$, then use $j(z)$ and $\wp(z)$. This result is obtained by the theory of complex multiplication.

In order to describe a finite Abelian extension of a number field, one should use the notion of conductor which controls the ramification. Class field theory gives a complete description of Abelian extensions and structure of their Galois group. Let K be a number field and let $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ be a modulus. The ray class field $K_\mathfrak{m}$ is an Abelian extension of K with Galois group

$$\mathrm{Gal}(K_\mathfrak{m}/K) \cong I_K(\mathfrak{m}_0)/P_{K,1}(\mathfrak{m})$$

A prime ideal $\mathfrak{p}$ of K ramifies in $K_\mathfrak{m}$ if and only if $\mathfrak{p}$ divides $\mathfrak{m}_0$.

The Dirichlet L-series gives us a bridge between the Galois group of cyclotomic extensions and the analytic theory. We can generalize this idea and consider

$$\chi : I_K(\mathfrak{m}_0)/P_{K,1}(\mathfrak{m}) \to \mathbf{C}^*$$

We define the generalized Dirichlet L-series (or Hecke L-series) by

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs through non-zero ideals of $\mathcal{O}_K$. We put $\chi(\mathfrak{a}) = 0$ if $(\mathfrak{a}, \mathfrak{m}_0) \neq 1$. Since Dirichlet L-series gives crucial information about Abelian extensions of $\mathbf{Q}$, we can expect the same from Hecke L-series for the corresponding extensions.

Conjecture 4.2 (Stark)**.** There exists elements $\pi(\sigma) \in K_\mathfrak{m}$ for each $\sigma \in G_\mathfrak{m} = \mathrm{Gal}(K_\mathfrak{m}/K)$ such that

$$L'(0,\chi) = -\frac{1}{W_K} \sum_{\sigma \in G_\mathfrak{m}} \chi(\sigma) \log ||\pi(\sigma)||$$

where $||\alpha|| = |\alpha|$ if K is totally real and $||\alpha|| = |\alpha|^2$ if K has only a single pair of complex embeddings.

Note that if $r_2 \geq 0$ then $L'(0,\chi) = 0$ and the conjecture is trivial. Stark uses the values of L-series at $s = 0$ because these are related with simpler formulas. For example

$$\lim_{s \to 0} \frac{\zeta_L(s)}{s^r} = -\frac{h_L \mathrm{Reg}(L)}{W_L}$$

which is much nicer than the class number formula at $s = 1$. Stark shows that his conjecture is true imaginary quadratic fields (the case of rational numbers was already known).

References

[1] Stein & Shakarchi, *Complex Analysis.*
[2] Neukirch, *Algebraic Number Theory.*
[3] Washington, *Introduction to Cyclotomic Fields.*
[4] Stark, L-Functions at s=1. IV. First Derivatives at s=0. *Adv. in Math. (1980).*