METU, Spring 2018, Math 523.

Homework 6 (due May 16)

1. (20 points) Let $K = \mathbf{Q}(\sqrt{30})$. Recall that we have shown that $\mathcal{O}_K$ is not a unique factorization domain. Find the structure of $\mathrm{Cl}(K)$.

2. (30 points) There are 9 imaginary quadratic fields $K = \mathbf{Q}(\sqrt{m})$ with class number one (this is a well known fact due to Baker, Heegner and Stark). The set of squarefree integers m giving such fields is

$$M = \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

   - Verify that the imaginary quadratic field $K = \mathbf{Q}(\sqrt{m})$ has class number one, for each integer $m \in M$.
   - (*Exercise 5.10. Marcus*) Let $-2000 < m < 0$ be a squarefree negative integer such that $K = \mathbf{Q}(\sqrt{m})$ has class number one. Show that $m \in M$.

3. (20 points) Show that the following Diophantine equation has infinitely many solutions:

$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

   Give a procedure to obtain each solution. (Hint: Consider $K = \mathbf{Q}(\sqrt[3]{2})$.)

4. (30 points) (*The simplest cubic fields*) Let a be an integer such that $p = a^2 + 3a + 9$ is a prime number. Set $f(x) = x^3 - ax^2 - (a+3)x - 1$. Let ρ be a root of f and $K = \mathbf{Q}(\rho)$. (You can fix $a = 1$ for simplicity.)

   (a) Show that f is irreducible and compute d_K.

   (b) Is the extension $K/\mathbf{Q}$ normal? How many real embeddings are there?

   (c) Prove that $u = \rho$ and $v = \rho + 1$ are units in $\mathcal{O}_K$.

   (d) Find generators for the unit group $\mathcal{O}_K^\times$.