METU, Spring 2018, Math 523.

Homework 5 (due May 2)

1. (20 pts) Consider the cyclotomic field $K = \mathbf{Q}(\zeta_{12})$. Find the ideal prime decomposition of $p\mathcal{O}_K$ for $p \in \{2, 3, 5, 7, 13\}$.

2. (20 pts) Find a quadratic extension L of $K = \mathbf{Q}(\sqrt{-5})$ such that no prime ramifies in the extension L/K.

3. (30 pts) Let $L = \mathbf{Q}(\sqrt{2}, \sqrt{3})$. You are given that $\mathcal{O}_L = \mathbf{Z}[\alpha]$ where $\alpha = (\sqrt{2} + \sqrt{6})/2$.

   - Show that $L/\mathbf{Q}$ is normal and $\mathrm{Gal}(L/\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Use the Galois correspondence to sketch a diagram of subfields.
   - Find the prime ideal decomposition of $p\mathcal{O}_L$ for each prime $p \in \{2, 3, 5\}$. Determine the inertial degree and ramification index for each case.
   - Fix a prime ideal of $\mathfrak{P} \subset L$ lying over $3\mathbf{Z}$ and give its generators. If $K_1 = \mathbf{Q}(\sqrt{2}), K_2 = \mathbf{Q}(\sqrt{3})$ and $K_3 = \mathbf{Q}(\sqrt{6})$, then determine $\mathfrak{p}_i = \mathfrak{P} \cap K_i$ for each $i \in \{1, 2, 3\}$ and give its generators.

4. (30 pts) Set $\zeta = \exp(2\pi i/23)$. Consider $L = \mathbf{Q}(\zeta)$ and $K = (\sqrt{-23})$. Recall that $K \subset L$ and $\mathcal{O}_K = \mathbf{Z}[\theta]$ with $\theta = (1 + \sqrt{-23})/2$. Take $\mathfrak{p} = \langle 2, \theta \rangle \subset \mathcal{O}_K$. Let $\mathfrak{P} \subset \mathcal{O}_L$ be a prime ideal lying over $\mathfrak{p}$.

   - Show that $f(\mathfrak{P}|\mathfrak{p}) = 11$. Conclude that $\mathfrak{P} = \langle 2, \theta \rangle \subset \mathcal{O}_L$.
   - Show that $\mathfrak{p}$ is not principal in $\mathcal{O}_K$ whereas $\mathfrak{p}^3 = (\theta - 2)$.
   - Show that $\mathfrak{P}$ is not principal.
   - Show that if 2 is irreducible in the ring $\mathbf{Z}[\zeta]$.
   - Verify that the product

     $$(1 + \zeta^2 + \zeta^4 + \zeta^5 + \zeta^6 + \zeta^{10} + \zeta^{11}) \cdot (1 + \zeta + \zeta^5 + \zeta^6 + \zeta^7 + \zeta^9 + \zeta^{11})$$

     is divisible by 2 in the ring $\mathbf{Z}[\zeta]$.
   - Show that $\mathbf{Z}[\zeta]$ is not a UFD.