METU, Spring 2018, Math 523.

Homework 4 (due April 18)

1. Let K be a number field of degree n over $\mathbf{Q}$. Show that every non-zero ideal $\mathfrak{a} \subset \mathcal{O}_K$ is a free abelian group of rank n.

2. Consider the ideal $I = \langle x, 6 \rangle$ of the ring $R = \mathbf{Z}[x]$. Show that there are infinitely many prime ideals of R that is contained in I. Is it possible to represent I as a product of prime ideals? Is R integrally closed in its field of fractions? Is R Noetherian? Is R a Dedekind domain?

3. Suppose that $K = \mathbf{Q}(\sqrt{-23})$. Consider the ideal $\mathfrak{a} = \langle 3, 1 + \sqrt{-23} \rangle$ of $\mathcal{O}_K$. Show that $\mathfrak{a} \neq \langle 1 \rangle$. Show that $N(\mathfrak{a}) = 3$. Is $\mathfrak{a}$ principal? What about $\mathfrak{a}^2$ and $\mathfrak{a}^3$?

4. Set $\alpha = \sqrt[3]{19}$ and consider the number field $K = \mathbf{Q}(\sqrt[3]{19})$. Recall that $[\mathcal{O}_K : \mathbf{Z}[\alpha]] \neq 1$. Find the ideal prime decomposition of $\langle p \rangle \subset \mathcal{O}_K$ for $p \in \{2, 3, 5, 7\}$.

5. Show that the polynomial $x^4 + 1$ is reducible in $\mathbf{F}_p[x]$ for each prime $p \in \mathbf{Z}$.