METU, Spring 2018, Math 523.

Homework 3 (due March 28)

1. Show that every quadratic number field K is contained in a cyclotomic field $\mathbf{Q}(\zeta_n)$ for some positive integer n. Recall that we have shown $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer m. What is the relation between m and n, if n is assumed to be the minimal such integer? (Hint: See Marcus Chap. 2, Ex. 8).

2. Let K be the m-th cyclotomic field $\mathbf{Q}(\zeta_m)$. Recall that $\mathcal{O}_K = \mathbf{Z}[\zeta_m]$.

   (a) Let m be a power of a prime p. If $\gcd(p, ab) = 1$, then show that $(1-\zeta_m^a)/(1-\zeta_m^b)$ is a unit in $\mathcal{O}_K$.

   (b) Let $m = pq$ where p and q are distinct primes. Show that $1 - \zeta_m$ is a unit in $\mathcal{O}_K$.

3. Show that the Diophantine equation $x^2 - 6y^2 = 523$ has infinitely many solutions.

4. Show that $\mathbf{Z}[\sqrt{-30}\,]$ and $\mathbf{Z}[\sqrt{30}\,]$ are not unique factorization domains.

5. Show that the only solution of the Diophantine equation $x^3 = y^2 + 1$ is $(0, 1)$.