METU, Spring 2018, Math 523.

Homework 2 (due March 14)

1. **(10pt)** Show that $\mathbf{Q}(\sqrt{2}, i) = \mathbf{Q}(\zeta_8)$ where $\zeta_8 = \exp(2\pi i/8)$. Find all embeddings (monomorphisms) from $\mathbf{Q}(\sqrt{2}, i)$ to $\mathbf{C}$. Express each such embedding in the form $\sigma_i(\zeta_8) = \zeta_8^i$ for some $i \in \{1, 2, \ldots, 8\}$.

2. **(10pt)** Find a 6×6 matrix M with coefficients from $\mathbf{Z}$ such that the minimal polynomial of $\alpha = \sqrt[3]{3} + \sqrt{2}$ over $\mathbf{Q}$ is given by the determinant of $xI - M$. Find the minimal polynomial of α over $\mathbf{Q}$. Is α a unit in the ring $\mathbf{Z}[\alpha]$?

3. **(10pt)** Show that the additive group of the ring $\mathbf{Z}[\sqrt{5}/2]$ is not finitely generated.

4. **(10pt)** Show that $f(x) = x^3 + 5x + 3$ is irreducible. Let α be a root of $f(x)$ and let $K = \mathbf{Q}(\alpha)$.
   - Calculate $T_{\mathbf{Q}}^K(\alpha^i)$ for $i \in \{0, 1, 2, 3\}$.
   - Calculate $N_{\mathbf{Q}}^K(\alpha - j)$ for $j \in \{0, 1, 2\}$.

5. **(20pt)** Suppose that $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$. Show that $\mathcal{O}_K \neq \mathbf{Z}[\sqrt{2}, \sqrt{3}\,]$.

6. **(20pt)** Suppose that $K = \mathbf{Q}(\sqrt[3]{19})$. Find an integral basis.

7. **(20pt)** Suppose that α is a complex number such that $\alpha^5 = \alpha + 1$. If $K = \mathbf{Q}(\alpha)$ then show that $\mathcal{O}_K = \mathbf{Z}[\alpha]$.