METU, Spring 2018, Math 523.

Exercise Set 10

1. Let $\mathfrak{P} \subset L$ be a prime lying over $\mathfrak{p} \subset K$. Suppose that L/K is a normal extension with Galois group $G = \mathrm{Gal}(L/K)$. Show that

   - the decomposition group $D(\mathfrak{P}|\mathfrak{p})$ is a subgroup of G.
   - the inertia group $I(\mathfrak{P}|\mathfrak{p})$ is a normal subgroup of $D(\mathfrak{P}|\mathfrak{p})$.

2. Let $K = \mathbf{Q}(\zeta_m)$, the m-th cyclotomic field. Fix a prime $p \in \mathbf{Z}$. Write $m = p^k n$ for some integer n with $p \nmid n$. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal lying over p. Set $e = e(\mathfrak{p}|p)$ and $f = f(\mathfrak{p}|p)$. Prove that $\phi(m) = r \cdot e \cdot f$ with $e = \phi(p^k)$ and f is the multiplicative order of p mod n.

3. If $K = \mathbf{Q}(\zeta_m)$, then find the ideal prime decomposition of $p\mathcal{O}_K$ for $p \in \{2, 3, 5, 7\}$ and $m \in \{8, 12, 14, 15\}$.

4. If K is a number field in which no prime $p \in \mathbf{Z}$ ramifies, then prove that $K = \mathbf{Q}$.

5. Find a quadratic extension L of $K = \mathbf{Q}(\sqrt{-5})$ such that no prime ramifies in the extension L/K.

6. Let K be a number field. Prove that infinitely many primes $p \in \mathbf{Z}$ split completely in K, i.e. split into $[K : \mathbf{Q}]$ distinct factors.