METU, Spring 2018, Math 523.

Exercise Set 8

1. Let K be a number field and let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal. If α is a nonzero element of $\mathfrak{a}$, then show that $N(\mathfrak{a})$ divides $N_{\mathbf{Q}}^K(\alpha)$.

2. Let $\alpha = \sqrt{-5}$ and $K = \mathbf{Q}(\alpha)$. Suppose that $\mathfrak{a} = (120, 11\alpha - 19) \subset \mathcal{O}_K$.
   - Find all primes $p \in \mathbf{Z}$ such that $(\mathfrak{a} \cap \mathbf{Z}) \subset p\mathbf{Z}$.
   - Evaluate $N_{\mathbf{Q}}^K(11\alpha - 19)$. Is this integer related with $N(\mathfrak{a})$?
   - Find the ideal prime decomposition of $\mathfrak{a}$.

3. Consider the ideals $\mathfrak{a} = (2+\sqrt{-5})$ and $\mathfrak{b} = (3)$ in $\mathbf{Z}[\sqrt{-5}\,]$. Show that $\mathfrak{a}+\mathfrak{b} = (3, 1-\sqrt{5})$ and $\mathfrak{a} \cap \mathfrak{b} = (9, 3 - 3\sqrt{-5})$.

4. Consider the number fields $\mathbf{Q}, \mathbf{Q}(i), \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{-29}), \mathbf{Q}(\sqrt[3]{2}), \mathbf{Q}(\zeta_5)$. For each one of these number fields,
   - find all the ideals that contain the element 30,
   - find all the ideals with norm 18, if there is any.

5. Set $\alpha = \sqrt[3]{19}$ and consider the number field $K = \mathbf{Q}(\sqrt[3]{19})$. Recall that $[\mathcal{O}_K : \mathbf{Z}[\alpha]] \neq 1$. Find the ideal prime decomposition of $(p) \subset \mathcal{O}_K$ for $p \in \{2, 3, 5, 7\}$.

6. Show that the polynomial $x^4 + 1$ is reducible in $\mathbf{F}_p[x]$ for each prime $p \in \mathbf{Z}$.

7. Let $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{r}$ be prime ideals of number fields $K \subset L \subset M$ respectively. Show that the ramification index and inertial degree are multiplicative in towers.
   - $e(\mathfrak{r}|\mathfrak{p}) = e(\mathfrak{r}|\mathfrak{q})e(\mathfrak{q}|\mathfrak{p})$.
   - $f(\mathfrak{r}|\mathfrak{p}) = f(\mathfrak{r}|\mathfrak{q})f(\mathfrak{q}|\mathfrak{p})$.

8. Let α be a root of $f(x) = x^3 - x - 1$ and let $K = \mathbf{Q}(\alpha)$.
   - Show that f is irreducible over $\mathbf{Q}$. Find an integral basis for $\mathcal{O}_K$ and compute the discriminant d_K.
   - Consider the ideals $\mathfrak{p} = (23, \alpha - 10)$ and $\mathfrak{q} = (23, \alpha - 3)$ of $\mathcal{O}_K$. Show that $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals.
   - Verify that $(23) = \mathfrak{p}^2\mathfrak{q}$. Determine the ramification index and inertial degree for each prime.

9. Let p be an odd prime and let $K = \mathbf{Q}(\zeta_p)$ be the p-th cyclotomic field.
   - Show that $\mathcal{O}_K = \mathbf{Z}[\zeta_p]$.
   - Consider the principal ideal $\mathfrak{p} = (1 - \zeta_p) \subset \mathcal{O}_K$ and compute $N(\mathfrak{p})$. Is $\mathfrak{p}$ prime?
   - Find the ideal prime decomposition of $(p) \subset \mathcal{O}_K$.