METU, Spring 2018, Math 523.

Exercise Set 7

1. Let $\mathfrak{p}$ be a proper ideal of a commutative ring R with identity 1_R. Show that the following are equivalent:
   - For all elements $a, b \in R$, $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
   - For all ideals $\mathfrak{a}, \mathfrak{b}$ in R, $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

2. Let R be an integral domain. For nonzero ideals $I, J \subset R$, define the relation
$$I \sim J \iff \alpha I = \beta J \text{ for some nonzero } \alpha, \beta \in R.$$
   - Prove that $\sim$ is an equivalence relation.
   - Describe the corresponding equivalence class if I is principal.
   - Given an ideal I, suppose that there is an ideal J such that IJ is principal. Does this property make the set of ideal classes a group?

3. Let R be an integral domain. Prove that the followings are equivalent.
   - Every ideal is finitely generated.
   - Every ascending chain of ideals stabilizes (**A**scending **C**hain **C**ondition).
   - Every non-empty set $\mathcal{S}$ of ideals has a maximal member.

4. Let K be a number field of degree n over $\mathbf{Q}$. Show that every non-zero ideal $\mathfrak{a} \subset \mathcal{O}_K$ is a free abelian group of rank n.

5. Consider the ideal $I = \langle x, 6\rangle$ of the ring $R = \mathbf{Z}[x]$. Show that there are infinitely many prime ideals of R that is contained in I. Is it possible to represent I as a product of prime ideals? Is R integrally closed in its field of fractions? Is R Noetherian? Is R a Dedekind domain?

6. Prove that a Dedekind domain is a unique factorization domain if and only if it is a principal ideal domain.

7. Consider the ideal $I = \langle x, p\rangle \subseteq \mathbf{Z}[x]$ for some prime element $p \in \mathbf{Z}$. Show that the ideal I^2 cannot be generated by 2 elements.

8. Let $K = \mathbf{Q}(\alpha)$ where $\alpha = \sqrt[3]{2}$. Recall that $\mathcal{O}_K = \mathbf{Z}[\alpha]$.
   - Consider the ideal $\langle 5\rangle \subset \mathcal{O}_K$. Verify that $\langle 5\rangle = \langle 5, \alpha + 2\rangle\langle 5, \alpha^2 + 3\alpha - 1\rangle$.
   - Set $\mathfrak{p} = \langle 5, \alpha^2 + 3\alpha - 1\rangle$. Show that there is an endomorphism of rings
$$\mathbf{Z}[x]/(5, x^2 + 3x - 1) \twoheadrightarrow \mathcal{O}_K/\mathfrak{p}.$$
   - Find an element $\beta \in \mathcal{O}_K$ such that $\langle 5, \alpha + 2\rangle^2 = \langle 25, \beta\rangle$.