METU, Spring 2018, Math 523.

Exercise Set 6

1. Show that the Diophantine equation $x^2 - 6y^2 = 523$ has infinitely many solutions.
2. Let $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer m. Find all units in $\mathcal{O}_K$ for each $m < 0$.
3. Show that $\mathbf{Z}[\sqrt{-30}\,]$ and $\mathbf{Z}[\sqrt{30}\,]$ are not a unique factorization domains.
4. Let R be a unique factorization domain and let d be a nonzero element of R. Show that there are only a finite number of distinct principal ideals that contain the element d. As an example, find all of ideals of $\mathbf{Z}[i]$ that contain 30.
5. If $K = \mathbf{Q}(\zeta_5)$ or $K = \mathbf{Q}(\zeta_8)$, then show that $\mathcal{O}_K$ is Euclidean.
6. Let R be a Euclidean domain. Show that $u \in R$ is a unit if and only if $N(u) = N(1_R)$.
7. Show that the only solution of the Diophantine equation $x^3 = y^2 + 1$ is $(0, 1)$.
8. Consider the Diophantine equation $x^3 = y^2 + 26$. Imitate the idea of the previous question by assuming $\mathbf{Z}[\sqrt{-26}\,]$ is a UFD. End up with the conclusion that there is no solution, a contradiction since $(3, \pm 1)$ and $(35, \pm 207)$ are solutions. Conclude that $\mathbf{Z}[\sqrt{-26}\,]$ is not a UFD.