METU, Spring 2018, Math 523.

Exercise Set 5

1. Let $K = \mathbf{Q}(\sqrt{3})$ and let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}_K$ that contains 2. Show that $\mathfrak{m} = \langle 2, 1+\sqrt{3}\rangle$ and $\mathfrak{m}^2 = \langle 2\rangle$.

2. Let d_K be the discriminant of the quadratic number field K. If $w = (\sqrt{d_K} + d_K)/2$, then show that $\{1, w\}$ is an integral basis for $\mathcal{O}_K$.

3. Give a pair of quadratic number fields K and L such that $\mathcal{O}_{KL} \neq \mathcal{O}_K\mathcal{O}_L$. If $d_K = \mathrm{disc}(\mathcal{O}_K)$ and $d_L = \mathrm{disc}(\mathcal{O}_L)$, then what is $\gcd(d_K, d_L)$ for the example you have given?

4. Let K and L be number fields with discriminants d_K and d_L. Set $d = \gcd(d_K, d_L)$. The compositum KL is the smallest field that contains both K and L by definition. Suppose that $[KL : \mathbf{Q}] = [K : \mathbf{Q}] \cdot [L : \mathbf{Q}]$. Then show that

$$\mathcal{O}_{KL} \subseteq \frac{1}{d}\mathcal{O}_K\mathcal{O}_L.$$

   In particular if $d = 1$, then show that $\mathcal{O}_{KL} = \mathcal{O}_K\mathcal{O}_L$. (See Marcus, Chap. 2, Thm. 12.)

5. Show that every quadratic field K is contained in a cyclotomic field $\mathbf{Q}(\zeta_k)$ for some positive integer k. Recall that we have shown $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer m. What is the relation between m and k, if k is assumed to be the minimal such integer? (Hint: See Marcus Chap. 2, Ex. 8).

6. Let K be the m-th cyclotomic field $\mathbf{Q}(\zeta_m)$. Recall that $\mathcal{O}_K = \mathbf{Z}[\zeta_m]$.

   (a) Let m be a power of a prime p. If $\gcd(p, ab) = 1$, then show that $(1-\zeta_m^a)/(1-\zeta_m^b)$ is a unit in $\mathcal{O}_K$.

   (b) Let $m = pq$ where p and q are distinct primes. Show that $1 - \zeta_m$ is a unit in $\mathcal{O}_K$.