METU, Spring 2018, Math 523.

Exercise Set 4

1. Show that $f(x) = x^3 + 5x + 1$ is irreducible. Let α be a root of $f(x)$ and let $K = \mathbf{Q}(\alpha)$.
   - Calculate $T_{\mathbf{Q}}^{K}(\alpha^i)$ for $i \in \{0, 1, 2, 3\}$.
   - Calculate $N_{\mathbf{Q}}^{K}(\alpha - j)$ for $j \in \{0, 1, 2\}$.

2. This is from Marcus (Chap. 2, Ex. 28). Let $f(x) = x^3 + ax + b$, a and $b \in \mathbf{Z}$ and assume that f is irreducible over $\mathbf{Q}$. Let α be a root of f and $K = \mathbf{Q}(\alpha)$.
   (a) Show that $f'(\alpha) = -\dfrac{2a\alpha + 3b}{\alpha}$.
   (b) Show that $2a\alpha + 3b$ is a root of $\left(\dfrac{x - 3b}{2a}\right)^3 + a\left(\dfrac{x - 3b}{2a}\right) + b$. Use this to find $N(2a\alpha + 3b)$.
   (c) Show that $\mathrm{disc}(1, \alpha, \alpha^2) = -(4a^3 + 27b^2)$.
   (d) Suppose that $\alpha^3 = \alpha + 1$. Prove that $\{1, \alpha, \alpha^2\}$ is an integral basis for $\mathcal{O}_K$. Do the same if $\alpha^3 + \alpha = 1$.

3. There is no primitive element theorem for the ring of integers. To see this, consider $K = \mathbf{Q}(\sqrt{7}, \sqrt{10})$. Show that $\mathcal{O}_K \neq \mathbf{Z}[\alpha]$ for all $\alpha \in \mathcal{O}_K$. For the details, see Marcus (Chap. 2, Ex. 30).

4. Let $\alpha = \sqrt[3]{m}$, where m is a cubefree integer. Find an integral basis for $K = \mathbf{Q}(\alpha)$. See Marcus (Chap. 2, Ex. 41).

5. Let $K = \mathbf{Q}(\sqrt{m}, \sqrt{n})$ where m and n are distinct squarefree integers other than 1. Find an integral basis for K. See Marcus (Chap. 2, Ex. 42).