METU, Spring 2018, Math 523.

Exercise Set 3

1. Show that $\mathbf{Q}(\sqrt{2}, i) = \mathbf{Q}(\zeta_8)$ where $\zeta_8 = \exp(2\pi i/8)$. Find all embeddings (monomorphisms) from $\mathbf{Q}(\sqrt{2}, i)$ to $\mathbf{C}$. Express each such embedding in the form $f_i(\zeta_8) = \zeta_8^i$.

2. For each of the following, determine if it is an algebraic integer or not.
   - $523\sqrt{5} + 2018$,
   - $(\sqrt{7} + 1)/2$,
   - $(\sqrt[3]{19^2} + \sqrt[3]{19} + 1)/3$,
   - $(\sqrt{2} + \sqrt{-1})/2$,
   - $(2\sqrt{-27} + 3)/6$

3. Find a 6×6 matrix M with coefficients from $\mathbf{Z}$ such that the minimal polynomial of $\alpha = \sqrt[3]{2} + \sqrt{5}$ over $\mathbf{Q}$ is given by the determinant of $xI - M$.

4. Show that the additive group of the ring $\mathbf{Z}[\sqrt{5}/2]$ is not finitely generated. Show that the additive group of the ring $\mathbf{Z}[(\sqrt{5} + 1)/2]$ is finitely generated.

5. Suppose that $K = \mathbf{Q}(\sqrt{m})$ for some squarefree integer $m \in \mathbf{Z}$. Prove that the subring of algebraic integers of K is given by

$$\mathcal{O}_K = \begin{cases} \{a + b\sqrt{m} : a, b \in \mathbf{Z}\} & if \;\; m \equiv 2, 3 \pmod 4, \\ \{(a + b\sqrt{m})/2 : a, b \in \mathbf{Z},\; a \equiv b \pmod 2\} & if \;\; m \equiv 1 \pmod 4. \end{cases}$$

6. Prove the Vandermonde determinant, i.e. show that

$$\det[t_i^{j-1}]_{n\times n} = \prod_{1\le i,j\le n} (t_i - t_j).$$

7. Show that $\sqrt{3} \notin \mathbf{Q}(\sqrt[4]{2})$. Show that $\sqrt{5} \in \mathbf{Q}(\zeta_5)$ where $\zeta_5 = \exp(2\pi i/5)$.