METU, Spring 2018, Math 523.

Exercise Set 2

1. We know that $R[x]$ is a principal ideal domain if R is a field (because the division algorithm works). What about the converse? If $R[x]$ is a principal ideal domain, what can you say about R? Is it necessarily a field?

2. Let R be a principal ideal domain. If $\mathfrak{p} \subset R$ is a prime ideal then prove that the quotient $R/\mathfrak{p}$ is also a principal ideal domain. Can you drop the primeness condition on the ideal $\mathfrak{p}$ and still have the same property?

3. Let R be a commutative ring with identity. The product of two ideals I and J consists of all finite sums of products ij where $i \in I$ and $j \in J$. Prove that

$$I \cdot J = I \cap J$$

if I and J are relatively prime, i.e. $I + J = R$. Is the above equality still true if we drop the condition of being relatively prime on ideals I and J?

4. Let R be a principal ideal domain and let A be an ideal of R. Show that A is prime if and only if A is maximal.

5. Let $K/\mathbf{Q}$ be a finite field extension and let α be a nonzero element in K. If $f(x) \in \mathbf{Q}[x]$ is a monic polynomial of smallest degree such that $f(\alpha) = 0$ then show that $f(x)$ is unique and irreducible.

6. Construct a field $\mathbf{F}_{16} \cong R/\mathfrak{m}$ with 16 elements by choosing a ring R and a maximal ideal $\mathfrak{m} \subset R$ suitably.
   - Determine an element $r \in R$ such that the corresponding element in $\mathbf{F}_{16}$ generates the multiplicative group $\mathbf{F}_{16}^{\times}$. Construct a subfield $\mathbf{F}_4$ of $\mathbf{F}_{16}$ with 4 elements.
   - What is $[\mathbf{F}_{16} : \mathbf{F}_4]$? Is there a nontrivial automorphism of $\mathbf{F}_{16}$ fixing the subfield $\mathbf{F}_4$ pointwise?

7. Let K be a field extension of $\mathbf{Q}$ such that $[K : \mathbf{Q}] = 2$. Show that $K = \mathbf{Q}[\sqrt{m}\,]$ for some integer $m \in \mathbf{Z}$. Give an example of a field extension $L/\mathbf{Q}$ such that $[L : \mathbf{Q}] = 3$ and $L \neq \mathbf{Q}[\sqrt[3]{m}\,]$ for any integer $m \in \mathbf{Z}$.

8. Let $R = \mathbf{Z}[\sqrt{-3}\,]$ and let $I = \langle 2, 1 + \sqrt{-3}\,\rangle$. Show that $I \neq \langle 2 \rangle$ but $I^2 = 2I$. Conclude that ideals in R do not factor uniquely into prime ideals.

9. Determine all subspaces of the $\mathbf{Q}$-vector space $\mathbf{Q}^2$. Determine all submodules of the $\mathbf{Z}$-module $\mathbf{Z}^2$.

10. Let R be an integral domain. Let $\pi \in R$ be a prime element and let $\alpha \in R$ be an irreducible element. What can you say about the ideals $\langle \pi \rangle$ and $\langle \alpha \rangle$ in terms of being prime and maximal?