METU, Spring 2018, Math 523.

Exercise Set 1

1. Consider the ring $R = \mathbf{Z}[\sqrt{-1}\,]$ with the norm map $N(a+bi) = a^2 + b^2$. This ring is referred as Gaussian integers.

   - Verify that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in R$.
   - Find all units of R.
   - Classify prime elements of R in terms of primes of $\mathbf{Z}$.
   - Let n be a positive integer. Show that n can be expressed as a sum of two square if and only if $n = N(\alpha)$ for some $\alpha \in R$. Classify all integers n which can be written as a sum of two squares.
   - Write a Pari/GP script to determine the number of ways you can express $n = 19890 = 2 \cdot 5 \cdot 9 \cdot 13 \cdot 17$ as a sum of two squares. Explain this number by using the properties of Gaussian integers.

2. Consider the ring $R = \mathbf{Z}[\sqrt{-2}\,]$.

   - Show that R is a Euclidean domain under the norm map $a + b\sqrt{-2} \mapsto a^2 + 2b^2$.
   - Find all units of R.
   - Classify prime elements of R in terms of primes of $\mathbf{Z}$.
   - Let p be an odd prime. Show that the congruence $x^2 \equiv -2 \pmod{p}$ has a solution if and only if $p \equiv 1, 3 \pmod{8}$. In other words show that the Legendre symbol satisfies $\left(\frac{-2}{p}\right) = 1$ if and only if $p \equiv 1, 3 \pmod{8}$. (Hint: use Gauss's lemma for quadratic reciprocity.)
   - Find all primes $p \in \mathbf{Z}$ of the form $p = x^2 + 2y^2$ for some $x, y \in \mathbf{Z}$.
   - Find all solutions of the equation $x^2 + 2y^2 = z^2$ with $x, y, z \in \mathbf{Z}$.

3. Consider the ring $R = \mathbf{Z}[\sqrt{-5}\,]$.

   - Show that R is **not** a Euclidean domain under the norm map $a + b\sqrt{-5} \mapsto a^2 + 5b^2$. Do this in two different ways:
     - Divide $1 + \sqrt{-5}$ by 2.
     - Show that $\mathbf{Z}[\sqrt{-5}\,]$ is not a UFD
   - Give an example of an irreducible element $\alpha \in R$ that is not prime.
   - Determine the primes p for which $\left(\frac{-5}{p}\right) = 1$ by using the quadratic reciprocity law.
   - Write a Pari/GP script to determine the primes $p < 1000$ which are of the form $p = x^2 + 5y^2$ for some $x, y \in \mathbf{Z}$. Do the same for $p = 2x^2 + 2xy + 3y^2$.
   - Find all solutions of the equation $x^2 + 5y^2 = z^2$ with $x, y, z \in \mathbf{Z}$.