Name:
Signature:

METU, Spring 2018, Math 523.

Midterm

1. (20pts) Let K and L be number fields with ring of integers $\mathcal{O}_K$ and $\mathcal{O}_L$, discriminants d_K and d_L, respectively. Determine for each of the following statements if it is true or not.

- $\mathcal{O}_{KL} = \mathcal{O}_K\mathcal{O}_L$.

- $K \cap L = \mathbf{Q}$ if and only if $\gcd(d_K, d_L) = 1$.

- If $\mathcal{O}_K = \mathbf{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$, then $K = \mathbf{Q}(\alpha)$.

- If $K = \mathbf{Q}(\alpha)$ for some $\alpha \in \mathcal{O}_K$, then $\mathcal{O}_K = \mathbf{Z}[\alpha]$.

2. (20pts) Let $f(x) = x^3 + ax + b$ be an irreducible polynomial over $\mathbf{Q}$. If α is a root of $f(x)$ and $K = \mathbf{Q}(\alpha)$, then determine the following traces and norms in terms of a and b.

- $T^K_{\mathbf{Q}}(1)$

- $T^K_{\mathbf{Q}}(\alpha)$

- $T^K_{\mathbf{Q}}(\alpha^2)$

- $T^K_{\mathbf{Q}}(\alpha^3)$

- $N^K_{\mathbf{Q}}(2)$

- $N^K_{\mathbf{Q}}(\alpha)$

- $N^K_{\mathbf{Q}}(\alpha^2)$

- $N^K_{\mathbf{Q}}(\alpha - 2)$

3. (20pts) Consider the polynomial $f(x) = x^4 + x + 1$ with integer coefficients.

- Show that $f(x)$ is irreducible in $\mathbf{F}_2[x]$ (and therefore in $\mathbf{Z}[x]$).

- Let $\alpha \in \mathbf{C}$ be a root of $f(x)$. Show that $\operatorname{disc}(1, \alpha, \alpha^2, \alpha^3) = 229$. Find the ring of integers of the number field $\mathbf{Q}(\alpha)$.

4. (20pts) For each of the following, determine if the given factorizations are distinct in the indicated rings.

- The factorizations $6 = 2 \cdot 3$ and $6 = (\sqrt{7} + 1)(\sqrt{7} - 1)$ in the ring $\mathbf{Z}[\sqrt{7}]$.

- The factorizations $6 = 2 \cdot 3$ and $6 = (6 - \sqrt{30})(6 + \sqrt{30})$ in the ring $\mathbf{Z}[\sqrt{30}]$.

5. (20pts) Suppose that $\zeta_p = \exp(2\pi i/p)$, a primitive p-th root of unity.

- Find coefficients $c_i \in \mathbf{Z}$ such that $\sqrt{5} = \sum_{i=0}^{3} c_i \zeta_5^i$. (Hint: Find $\min\left(\zeta_5 + \frac{1}{\zeta_5}, \mathbf{Q}\right)$).

- Show that $\sqrt{7}$ is not an element of $\mathbf{Q}(\zeta_7)$.