METU, Fall 2011, Math 523.

Homework 6

1. Let K be a number field and let $\mathfrak{a} \subset \mathcal{O}_K$ be a nonzero ideal. Recall that we define the norm of $\mathfrak{a}$ to be the finite index $N(\mathfrak{a}) = |\mathcal{O}_K/\mathfrak{a}|$.
   - Prove that $N(\mathfrak{a})$ divides $N_{\mathbf{Q}}^{K}(\alpha)$ for all $\alpha \in \mathfrak{a}$.
   - Prove that $N(\mathfrak{a}) = |N_{\mathbf{Q}}^{K}(\alpha)|$ if $\mathfrak{a} = (\alpha)$.

2. Let $\alpha = \sqrt{-5}$ and $K = \mathbf{Q}(\alpha)$. Suppose that $\mathfrak{a} = (120, 11\alpha - 19) \subset \mathcal{O}_K$.
   - Find all primes $p \in \mathbf{Z}$ such that $(\mathfrak{a} \cap \mathbf{Z}) \subset p\mathbf{Z}$.
   - Evaluate $N_{\mathbf{Q}}^{K}(11\alpha - 19)$. Is this integer related with $N(\mathfrak{a})$?
   - Find the ideal prime decomposition of $\mathfrak{a}$ in the Dedekind domain $\mathcal{O}_K$.

3. Let $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{r}$ be prime ideals of number fields $K \subset L \subset M$ respectively. Show that the ramification index and inertial degree are multiplicative in towers.
   - $e(\mathfrak{r}|\mathfrak{p}) = e(\mathfrak{r}|\mathfrak{q})e(\mathfrak{q}|\mathfrak{p})$.
   - $f(\mathfrak{r}|\mathfrak{p}) = f(\mathfrak{r}|\mathfrak{q})f(\mathfrak{q}|\mathfrak{p})$.

4. Let α be a root of $f(x) = x^3 - x - 1$ and let $K = \mathbf{Q}(\alpha)$.
   - Show that f is irreducible over $\mathbf{Q}$. Find an integral basis for $\mathcal{O}_K$ and compute the discriminant d_K.
   - Consider the ideals $\mathfrak{p} = (23, \alpha - 10)$ and $\mathfrak{q} = (23, \alpha - 3)$ of $\mathcal{O}_K$. Show that $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals.
   - Verify that $(23) = \mathfrak{p}^2\mathfrak{q}$. Determine the ramification index and inertial degree for each prime.

5. Let p be an odd prime and let $K = \mathbf{Q}(\zeta_p)$ be the p-th cyclotomic field.
   - Show that $\mathcal{O}_K = \mathbf{Z}[\zeta_p]$.
   - Consider the principal ideal $\mathfrak{p} = (1 - \zeta_p) \subset \mathcal{O}_K$ and compute $N(\mathfrak{p})$. Is $\mathfrak{p}$ prime?
   - Find the ideal prime decomposition of $(p) \subset \mathcal{O}_K$.