METU, Spring 2012, Math 515.

Homework 2

(due March 19)

Do not forget stapling your homework!

1. Let A be a ring with nilradical $\mathfrak{N}$ and Jacobson radical $\mathfrak{R}$.
   - Show that $x \in \mathfrak{R} \Leftrightarrow 1 - xy \in A^\times$ for all $y \in A$.
   - If A is a finite ring, then show that $\mathfrak{N} = \mathfrak{R}$. (Hint: For each $y \in A$, show the existence of an element $e = y^k \in A$ such that $e^2 = e$. Can $1 - e$ be a unit in A?)

2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a ring A. Is the following statement true for positive integers m and n? Prove or disprove.

$$\mathfrak{a} + \mathfrak{b} = (1) \Leftrightarrow \mathfrak{a}^n + \mathfrak{b}^m = (1).$$

3. Let $\mathcal{O} = \mathbf{Z}[\sqrt{-5}]$ and consider the ideals $\mathfrak{p} = (2, 1 + \sqrt{-5})$ and $\mathfrak{q} = (3, 1 - \sqrt{-5})$. In this question, please do not use technical tools from Math 523.
   - Show that $\mathfrak{p}$ and $\mathfrak{q}$ are not principal ideals. Are they maximal?
   - Describe the ideals $\mathfrak{p} + \mathfrak{q}$, $\mathfrak{p} \cap \mathfrak{q}$, $(\mathfrak{p} : \mathfrak{q})$, $(\mathfrak{q} : \mathfrak{p})$ by giving generators. For each ideal check if it is principal, or not.
   - Consider the set $S = \{\alpha \in \mathcal{O} : \alpha - 1 \in \mathfrak{p}, \alpha - 2\sqrt{-5} \in \mathfrak{q}\}$. Give a condition on integers x and y so that $x\sqrt{-5} + y$ is an element of S. Is your condition necessary for $x\sqrt{-5} + y$ to be in S?

4. If $T : V \to V$ is a linear transformation on a vector space V over a field $\mathbf{F}$, then V can be made into an $\mathbf{F}[x]$-module by setting $xv = Tv$ for any $v \in V$. For each of the following transformations T, find all $\mathbf{F}[x]$-submodules of V.
   - $V = \mathbf{R}^2$ and $T(x, y) = (-y, x)$.
   - $V = \mathbf{R}^2$ and $T(x, y) = (0, y)$.
   - $V = \mathbf{R}^3$ and $T(x, y, z) = (z, x, y)$.

5. An element m of the A-module M is called a torsion element if $xm = 0$ for some non-zero $a \in A$. The set of torsion elements in M is denoted by $\mathrm{Tor}(M)$.
   - If A is an integral domain then prove that $\mathrm{Tor}(M)$ is a submodule of M. Give an example so that $\mathrm{Tor}(M)$ is not a submodule.
   - Show that if A has zero divisors than every non-zero A-module has torsion elements.