Name, Surname:

METU, Spring 2012.
Math 515, Final Exam.
May 25, 9:00, 120 minutes.

Choose 6 six questions to answer. **Indicate clearly** which questions you have chosen. Each question is of equal worth.

1. Give the definition of nilradical $\mathfrak{N}$ of a ring A. Prove that $\mathfrak{N}$ is the intersection of all the prime ideals of A.
2. Let $f : A \to B$ be a ring homomorphism. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A, B respectively. Give the definitions of $\mathfrak{a}^e$ and $\mathfrak{b}^c$ (extended and contracted). Determine for each of the following statements if it is TRUE or FALSE. If true give a proof, otherwise provide a counterexample.
   - For any $\mathfrak{b}$, there exists $\mathfrak{a}$ such that $\mathfrak{b} = \mathfrak{a}^e$.
   - $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$
   - If $\mathfrak{b}$ is prime then $\mathfrak{b}^c$ is prime.
3. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals. If an ideal $\mathfrak{a}$ is contained in $\bigcup_{i=1}^n \mathfrak{p}_i$ then show that $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.
4. Let M, N be A-modules. Show that $M \otimes_A N$ and $N \otimes_A M$ are isomorphic as A-modules.
5. Show that the operation S^{-1} is exact (i.e. if $M' \to M \to M''$ is exact at M, then show that $S^{-1}M' \to S^{-1}M \to S^{-1}M''$ is exact at $S^{-1}M$).
6. Let $A \subseteq B$ be integral domains, B integral over A. Then show that B is a field if and only A is a field.
7. Determine for each of the following statements if it is TRUE or FALSE. If true give a proof, otherwise provide a counterexample.
   - The power of a prime ideal is primary.
   - The power of a maximal ideal is primary.
8. Prove that M is a Noetherian A-module if and only if every submodule of M is finitely generated.
9. Give the definition of an irreducible ideal. In a Noetherian ring A, show that every ideal is a finite intersection of irreducible ideals.
10. Let A be an Artin ring. Prove that every prime ideal of A is maximal and A has only a finite number of maximal ideals.