M E T U - Department of Mathematics

Math 473 - Ideals, Varieties and Algorithms

Fall 2018 Ö. Küçüksakallı	Midterm 2 December 13, 17:40 100 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 points) State the Dickson's Lemma.

Prove that Dickson's Lemma is equivalent to the following statement: given a nonempty subset $A \subseteq \mathbb{Z}^n_{\geqslant 0}$, there are finitely many elements $\alpha(1), \ldots, \alpha(s) \in A$ such that for every $\alpha \in A$, there exists some $i \in \{1, \ldots, s\}$ and some $\gamma \in \mathbb{Z}^n_{\geqslant 0}$ such that $\alpha = \alpha(i) + \gamma$.

Question 2 (25 points) State the definition of Gröbner basis. Explain briefly why a Gröbner basis always exists.

Let G and H be Gröbner bases for the same ideal $I \subseteq k[x_1, x_2, \ldots, x_n]$ for a fixed monomial ordering. Show that $\overline{f}^G = \overline{f}^H$ for all $f \in k[x_1, x_2, \ldots, x_n]$.

Question 3 (25 points) Consider the following algorithm which looks like the Buchberger's algorithm. Note that there is a "minor" change.

$G = F$
REPEAT
    $G' = G$
    FOR each pair $\{p, q\}$, $p \neq q$ in G' DO
        $r = S(p, q)$
        IF $r \neq 0$ THEN $G = G \cup \{r\}$
UNTIL $G = G'$
RETURN G

Fix the monomial order to be the lexicographic order with $x > y$. If the input is $F = \{xy, x + y\}$, then show that this algorithm never terminates.

Question 4 (25 points) Consider the polynomial ring $\mathbb{Q}[x, y, z]$. Fix the monomial order to be the lexicographic order with $x > y > z$. Let $F = (xy + z, x + yz, y + z^2)$ and $I = \langle F \rangle$, the ideal generated by F.

(a) Show that the Buchberger's algorithm gives the output $G = F \cup \{-z^5 + z\}$ for the input F. (Omit the details if the division algorithm gives zero remainder.)

(b) Is F a Gröbner basis of I?

(c) What is the reduced Gröbner basis of I?