# M E T U - Department of Mathematics

## Math 473 - Ideals, Varieties and Algorithms

| Fall 2018<br>Ö. Küçüksakallı | Midterm 1<br>November 7, 17:40<br>100 minutes<br>4 questions on 4 pages. | Surname:<br>Name:<br>Student No:<br>Signature: |
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| 1 | 2 | 3 | 4 | | Total |
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**Question 1. (25 points)** Consider the following algorithm which is similar to the division algorithm in $k[x]$. Note that there is a "minor" change.

Input: $f, g \in k[x]$ & Output: $q, r \in k[x]$

$q := 0; r := f;$
WHILE $r \neq 0$ AND LT$(g)$ divides LT$(r)$ DO
    $q := q + (\mathrm{LT}(r)/\mathrm{LT}(g))g$
    $r := r - (\mathrm{LT}(r)/\mathrm{LT}(g))g$
RETURN $q, r$

**(a)** Find the output for $f = x^3 + x + 1$ and $g = x^2 - x$ in $\mathbf{Q}[x]$.

**(b)** Show that this algorithm terminates for each input.

**Question 2 (25 points)** Using the lexicographic order in $\mathbf{Q}[x, y, z]$ (with $x >_{\text{lex}} y >_{\text{lex}} z$), perform the multivariate division algorithm to divide $f = xz + x^2y^2$ by the indicated pair where $f_1 = x - y^2$ and $f_2 = xy - 1$.

**(a)** with $F = (f_1, f_2)$

**(b)** with $F = (f_2, f_1)$

**(c)** Is $f$ an element of the ideal $I = \langle f_1, f_2 \rangle$? (Hint: Plug in $x = y = z = 1$.)

**Question 3 (25 points)** Let $V \subseteq \mathbb{R}^3$ be the curve parametrized by $(t, t^3, t^4)$.

**(a)** Prove that V is an affine variety.

**(b)** Determine $\mathbf{I}(V)$, the ideal of $V$. (Hint: Express each element $f \in \mathbf{R}[x, y, z]$ in the form $f = (y - x^3)h_1 + (z - x^4)h_2 + r$.)

**Question 4 (25 points)** Derive a parametrization of the hyperbola $x^2 - 2y^2 = 1$, i.e. find $f(t), g(t) \in \mathbf{R}[t]$ such that $(f(t), g(t))$ gives all points on the hyperbola except possibly a point. (Hint: Consider the nonvertical lines through the point $P(-1, 0)$ on the hyperbola.)