M E T U - Department of Mathematics

Math 473 - Ideals, Varieties and Algorithms

Fall 2018 Ö. Küçüksakallı	Final January 18, 09:30 120 minutes 4 questions on 4 pages.	Surname: Name: Student No: Signature:

1	2	3	4		Total

Question 1. (25 points) Consider the polynomial ring $\mathbb{R}[x, y]$ under the lexicographic order with $x > y$. Set $f = x^4 - 2x^2 - y^4 + 2y^2 \in \mathbb{R}[x, y]$.

(a) Show that $G = \{x^2 - y^2, xy^2 - x, y^3 - y\}$ is a Gröbner basis for $I = \left\langle f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$.

(b) Find all singular points of the curve $f(x, y) = 0$.

Question 2 (25 points) (a) State the Extension Theorem for $\mathbb{C}[x_1, \ldots, x_n]$.

(b) Let f be a non-constant polynomial in $\mathbb{C}[x, y]$. Show that $\mathbf{V}(f)$ is infinite by using the Extension Theorem.

(c) Let f and g be non-constant polynomials in $\mathbb{C}[x, y]$. Show that $\mathbf{V}(f, g)$ is infinite if and only if f and g have a common irreducible factor in $\mathbb{C}[x, y]$.

Question 3 (25 points) (a) Write a pseudocode for the Buchberger's Algorithm.

(b) Explain briefly why this algorithm terminates.

(c) Explain the connection between the Buchberger's Algorithm and the Euclidean Algorithm for polynomials with one variable.

Question 4 (25 points) Consider the surface S parametrized by

$$\begin{aligned} x &= uv, \\ y &= u^2 - uv, \\ z &= v^2, \end{aligned}$$

with $u, v \in k$ where $\mathrm{char}(k) = 0$. Consider the ideal $I = \langle x - uv, y - u^2 + uv, z - v^2 \rangle$ of $k[u, v, x, y, z]$. You are given that the reduced Gröbner basis for I, under the lex order with $u > v > x > y > z$, is $G = \{g_1, \ldots, g_6\}$ where

$$\begin{aligned} g_1 &= u^2 - x - y, & g_4 &= uz - vx, \\ g_2 &= uv - x, & g_5 &= v^2 - z, \\ g_3 &= ux - vx - vy, & g_6 &= x^2 - xz - yz. \end{aligned}$$

(a) Is $\mathbf{V}(g_6)$ the smallest variety over $k = \mathbb{C}$ containing S?

(b) Show that the parametrization does not fill up the variety $\mathbf{V}(g_6)$ over $k = \mathbb{R}$.

(c) The pairs $(u, v) = (1, 1)$ and $(u, v) = (-1, -1)$ parametrize the same point $(1, 0, 1)$. Find all points $(x, y, z) \in S \cap \mathbb{C}^3$ parametrized by distinct pairs $(u, v) \in \mathbb{C}^2$.