Name and Surname:
Student Number:

Math 466 - Fall 2019 - METU

Midterm 2 - November 20 - 17:40 - 120 minutes

1. Consider $P = (1, 1, 1), Q = (-1, -1, 1), R = (1, -1, -1)$ and $S = (-1, 1, -1)$ and

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- Show that the points P, Q, R, S form a tetrahedron in $\mathbb{R}^3$.

  The distance between each pair of points P, Q, R, S is $2\sqrt{2}$. Any three points out of these four points form an equilateral triangle. There are four such equilateral triangles and they form a tetrahedron.

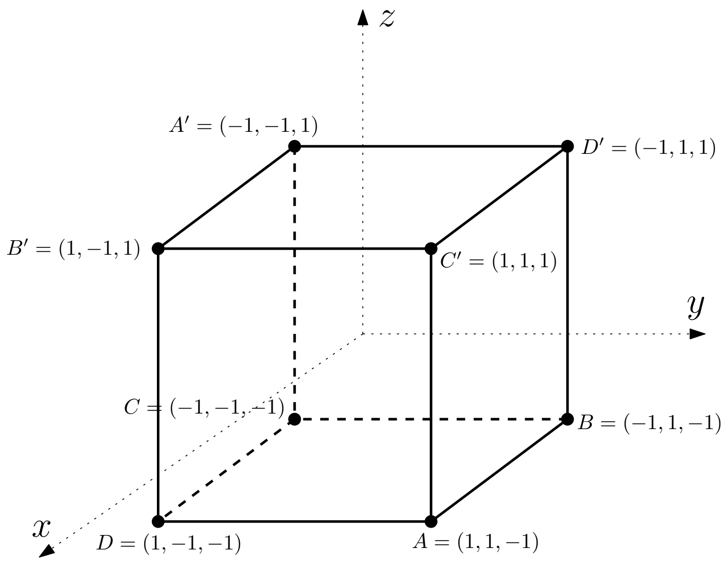
- Show that the group $G = \langle A, B \rangle$ acts on the set $X = \{P, Q, R, S\}$ by left multiplication. (Regard P, Q, R and S as column vectors.)

  We first verify that $AP = Q, AQ = P, AR = S, AS = R$ and $BP = P, BQ = R, BR = S, BS = Q$. For each $g \in G$ we have $gX = X$ because $AX = X$ and $BX = X$. We have $Ix = x$ for each point $x \in X$. Let g_1, g_2 be elements of G which are matrices. We have $g_1(g_2x) = (g_1g_2)x$ for each $x \in X$ by the associative property of the matrix multiplication. Thus G acts on X. As a result, there exists a homomorphism $\psi : G \to S_X = S_4$. In particular, $\psi(A) = (PQ)(RS)$ and $\psi(B) = (QRS)$.

- Write a group isomorphism between G and A_4.

  The points P, Q, R and S are not collinear, i.e. they do not lie on a single line. It follows that there is no nontrivial element of SO_3 that fix all the points in X. Thus the homomorphism ψ is injective. The image of ψ contains the even permutations $\psi(A) = (PQ)(RS)$ and $\psi(B) = (QRS)$ which are of order 2 and 3, respectively. The alternating group A_4 has no subgroup of order 6. We must have $\text{Im}(\psi) = \langle \psi(A), \psi(B) \rangle = A_4$. Therefore, the map $\psi : G \to A_4$ is a group isomorphism.

2. Consider the cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$ with rotational symmetry group G. Regard G as a subgroup of SO_3. Let S be the sphere centered at the origin with radius $\sqrt{3}$. Let X be the set of poles over S of nontrivial elements g in G.



- Show that $P = (\sqrt{3}, 0, 0) \in X$.

  The cube has a rotational symmetry around the x-axis through $\pi/2$ which induces the permutation $(AC'B'D)(BD'A'C)$. We regard this rotation as a matrix $R \in SO_3$. The axis of R intersects the sphere S at $(\pm\sqrt{3}, 0, 0)$ which are poles of G.

- The group G acts on X. Find the orbit $G(P)$.

  The faces of the cube are permuted by the rotational symmetries of the cube. As a result, the poles that lie over the y-axis and z-axis are in the same orbit as P. More precisely, we have

$$G(P) = \{(\pm\sqrt{3}, 0, 0), (0, \pm\sqrt{3}, 0), (0, 0, \pm\sqrt{3})\}$$

- Find the stabilizer group G_P.

  We have $G_P = \{R, R^2, R^3, I\}$ where R is the matrix from the first part.

- Verify the orbit-stabilizer formula $|G| = |G(x)| \cdot |G_x|$ with $x = P$.

  We verify that $|G| = 24 = 6 \cdot 4 = |G(P)| \cdot |G_P|$.

3. Consider the group $G = H \times_\varphi J$. What is the binary operation of this group? What is the identity element? What is the inverse of (h, j)?

- The binary operation is defined to be $(h_1, j_1)(h_2, j_2) = (h_1\varphi(j_1)(h_2), j_1 j_2)$.
- The identity element is (e_H, e_J) where e_H and e_J are the identity elements of H and J, respectively.
- The inverse of (h, j) is $(h, j)^{-1} = (\varphi(j)^{-1}(h^{-1}), j^{-1})$

As an example, consider $H = SO_2$ and $J = \mathbb{Z}_2$. Consider the group homomorphism $\varphi : J \to \mathrm{Aut}(H)$ where $\varphi(0)$ is the identity map $A \mapsto A$ and $\varphi(1)$ is the map $A \mapsto A^{-1}$ on H. Show that $H \times_\varphi J$ is isomorphic to the orthogonal group O_2.

Set $a = (A, 0)$ and $b = (I, 1)$. We have $b^{-1} = b$ and

$$\begin{aligned} bab^{-1} &= (I, 1)(A, 0)(I, 1) \\ &= (I\varphi(1)(A), 0 + 1)(I, 1) \\ &= (A^{-1}, 1)(I, 1) \\ &= (A^{-1}, 0) \\ &= a^{-1}. \end{aligned}$$

Fix $B \in O_2 \setminus SO_2$, an arbitrary reflection. For each rotation $A \in SO_2$, we have $BAB^{-1} = A^{-1}$. An arbitrary orthogonal matrix can be written uniquely in the form AB^ε for some $\varepsilon \in \{0, 1\}$. Similarly, an arbitrary element of $H \times_\varphi J$ can be written uniquely in the form $(a, \varepsilon) = ab^\varepsilon$ for some $\varepsilon \in \{0, 1\}$. We define $\psi : H \times_\varphi J \to O_2$ by the formula

$$\psi(ab^\varepsilon) = AB^\varepsilon.$$

This map is clearly bijective. We need to verify that it is a group homomorphism. The most important case that has to be verified is the following:

$$\psi(ba) = \psi(babb) = \psi(a^{-1}b) = A^{-1}B = BABB = BA = \psi(b)\psi(a).$$

The other cases can be verified easily.

Hint: Set $a = (A, 0)$ and $b = (I, 1)$ and compute bab^{-1}. Construct $\psi : H \times_\varphi J \to O_2$ which satisfies $\psi(a) = A$ and $\psi(b) = B$ for some suitable $B \in O_2 \setminus SO_2$.

4. For each of the following isometries, determine the type of the isometry; is it a translation, a rotation, a reflection or a glide reflection? If it is a rotation, then find its center and angle. If it is a reflection or a glide reflection, then find its mirror.

- $f = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$ or $f(x, y) = (y + 1, x + 1)$.

  The isometry f is a glide reflection. Its mirror is the line $y = x$. The reflection is followed by the translation $(x, y) \mapsto (x + 1, y + 1)$.

- $g = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$ or $g(x, y) = (-y, x - 1)$.

  The isometry g preserves the orientation and it is not a translation. It must be a rotation. Indeed, it is the rotation around the point $(1/2, -1/2)$ through the angle $\pi/2$.

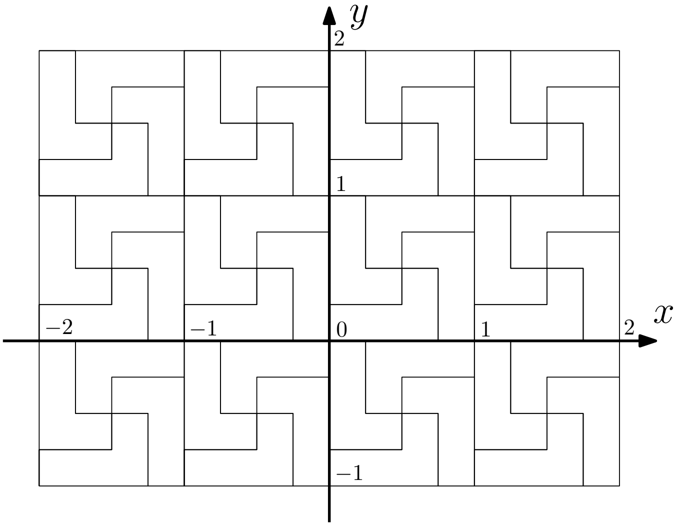
- fg.

  We first compute that $fg = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$ or $fg(x, y) = (x, 1 - y)$. The isometry fg is a reflection. Its mirror is the line $y = 1/2$.

- f^2.

  The isometry f^2 is a translation since $f^2 = \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ or $f^2(x, y) = (x + 2, y + 2)$.

5. For each of the following, let G be the group of isometries fixing the given pattern. Find generators for the translation subgroup H of G. Determine the lattice L and its type. Determine the structure of the point group $J = \pi(G)$.



The translation subgroup H is generated by

- $\tau_1(x, y) = (x + 1, y)$ and
- $\tau_2(x, y) = (x, y + 1)$.

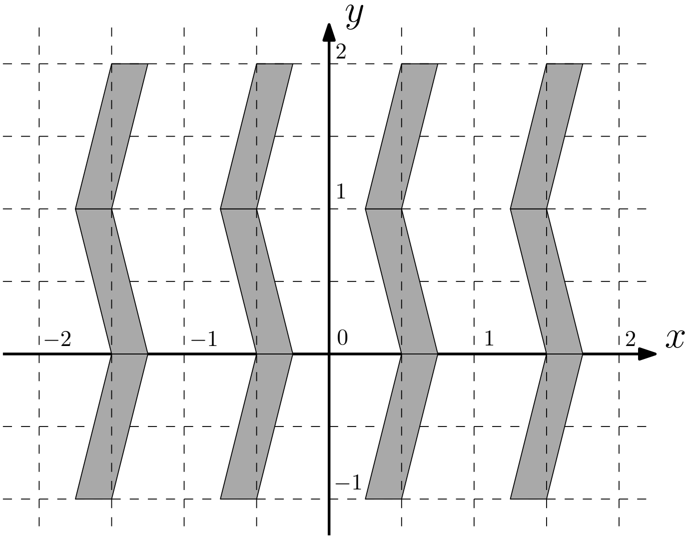
The lattice L is given by

$$L = \{(m, n) : m, n \in \mathbb{Z}\}$$

which is a square lattice.

The wallpaper pattern is preserved if we rotate around the origin through the angle $\pi/2$. There is no reflection. Thus we have

$$J = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong \mathbb{Z}_4.$$



In this case, the translation subgroup H is generated by

- $\tau_1(x, y) = (x + 1, y)$ and
- $\tau_2(x, y) = (x, y + 2)$.

The lattice L is given by

$$L = \{(m, 2n) : m, n \in \mathbb{Z}\}$$

which is a rectangular lattice.

The wallpaper pattern is preserved if we reflect about the x-axis. There is also a glide reflection preserving the pattern. Observe that the reflection about the vertical line $x = 1/2$ followed by the translation $(x, y) \mapsto (x, y + 1)$ preserves the pattern. Thus we have

$$J = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$